Multi-bubble nodal solutions to slightly subcritical elliptic problems with Hardy terms

Thomas Bartsch, Qianqiao Guo*

Abstract The paper is concerned with the slightly subcritical elliptic problem with Hardy term

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2-\varepsilon} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$, in dimensions $N \geq 7$. We prove the existence of multi-bubble nodal solutions that blow up positively at the origin and negatively at a different point as $\varepsilon \to 0$ and $\mu = \varepsilon^\alpha$ with $\alpha > \frac{N-4}{N-2}$. In the case of Ω being a ball centered at the origin we can obtain solutions with up to 5 bubbles of different signs. We also obtain nodal bubble tower solutions, i.e. superpositions of bubbles of different signs, all blowing up at the origin but with different blow-up order. The asymptotic shape of the solutions is determined in detail.

MSC 2010: Primary: 35B44; Secondary: 35B33, 35J60, 35J20, 35J70

Key words: Hardy term, critical exponent, slightly subcritical problems, nodal solutions, multi-bubble solutions, bubble towers, singular perturbation methods

1 Introduction

The paper is concerned with the semilinear singular problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2-\varepsilon} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 7$, is a smooth bounded domain with $0 \in \Omega$; $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent. We study the existence of nodal (i.e. sign changing) solutions that have multiple blow ups as $0 < \varepsilon \to 0$ and $\mu = \mu_0 \varepsilon^\alpha$, with $\mu_0 > 0$ and $\alpha > 0$ constants.

The blow-up phenomenon for positive and for nodal solutions to problem (1.1) has been studied extensively in the case $\mu = 0$. It was proved in [8, 18, 22, 27, 28] that as $\varepsilon \to 0^+$, the positive solution u_ε blows up and concentrates at a critical point of the Robin's function of Ω. In [3, 29], the existence of positive solutions with multiple bubbles was considered. In convex domains a positive solution cannot have multiple bubbles, see [20]. The existence of nodal solutions with k bubbles at k different points was

*Q.G. was supported by the National Natural Science Foundation of China (Grant Nos.11001221,11271299) and the Fundamental Research Funds for the Central Universities (3102015ZY069).

proved in [4] in the case $k = 2$, in [5] in the case $k = 4$ when Ω is convex and satisfies a certain symmetry, and in [6] in the case $k = 3$ when Ω is a ball. On the other hand, nodal solutions with tower of bubbles were obtained in [25, 26]. All these papers only treat the regular case $\mu = 0$.

When $\mu \neq 0$, the Hardy potential $\frac{1}{|x|^2}$ cannot be regarded as a lower order perturbation because it has the same homogeneity as the Laplace operator and does not belong to the Kato class. This makes the analysis much more complicated compared with the case $\mu = 0$. For the problem with Hardy type potentials and critical exponents, much attention has been paid to the existence of positive and nodal solutions, see e.g. [9, 10, 14, 17, 19, 21, 23, 30, 31, 33]. However, few results are known about the existence of positive or nodal solutions with multiple bubbles to the problem involving Hardy type potentials and critical exponents. We are only aware of the papers [15, 16], dealing with the problem

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2} u = k(x) u^{2^*-1}, \\ u \in D^{1,2}(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}; \end{cases} \tag{1.2}$$

here $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) | \, |\nabla u| \in L^2(\mathbb{R}^N)\}$. In [16] the existence of positive bubble tower solutions to (1.2), blowing up at the origin, was proved as $\epsilon \to 0$, when $k(x) = 1 + \varepsilon K(x)$ with $K(x)$ a continuous bounded function. These solutions, called fountain-like in [16], are superpositions of positive bubbles. In [15] the existence of a positive solution to (1.2) blowing up at a critical point of $k(x)$ was obtained as $\mu \to 0^+$. In [11] Cao and Peng investigated the asymptotic behavior of positive solutions to (1.1) in a ball.

In this paper we are interested in the existence of multi-bubble nodal solutions to problem (1.1) as $\varepsilon, \mu \to 0$. Compared with [15, 16] the location of the bubbles does not depend on the shape of a coefficient function $k(x)$ but on the subtle influence of the geometry of the domain. We obtain two types of solutions depending on the exponent α in the relation $\mu = \mu_0 \varepsilon^\alpha$ where $\mu_0 > 0$. If $\alpha > \frac{N-4}{N-2}$ we prove the existence of nodal solutions blowing up at different points, positively at the origin and negatively at other points. The bubble tower solutions exist for $\alpha = 1$, that is, when μ has the same order as ε when $\varepsilon \to 0^+$. In that case, on any smooth bounded domain, we obtain nodal solutions that are superpositons of bubbles with different signs, all blowing up at the origin. The proofs are based on the Lyapunov-Schmidt reduction scheme.

In order to state our results we introduce some notation. By Hardy's inequality, the norm

$$\|u\|_\mu := \left(\int_\Omega (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx \right)^{\frac{1}{2}}$$

is equivalent to the norm $\|u\|_0 = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}$ on $H_0^1(\Omega)$ provided $0 \leq \mu < \overline{\mu}$. This will of course be the case for $\mu = \mu_0 \varepsilon^\alpha$ with $\varepsilon > 0$ small. As in [17] we write $H_\mu(\Omega)$ for the Hilbert space consisting of $H_0^1(\Omega)$ functions with the inner product

$$(u, v) := \int_\Omega \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2} \right) dx.$$

It is known that the nonzero critical points of the energy functional

$$J_\varepsilon(u) := \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) dx - \frac{1}{2^* - \varepsilon} \int_\Omega |u|^{2^*-\varepsilon} dx$$

defined on $H_\mu(\Omega)$ are precisely the nontrivial weak solutions to problem (1.1).

Next we introduce two limiting problems. The first one is

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

It is well known that the nontrivial least energy (positive) solutions to (1.3) are the instantons

$$U_{\delta,\zeta} := C_0 \left(\frac{\delta}{\delta^2 + |x-\zeta|^2} \right)^{\frac{N-2}{2}}$$

with $\delta > 0$, $\zeta \in \mathbb{R}^N$ and $C_0 := (N(N-2))^{\frac{N-2}{4}}$, cf. [1, 32]. These solutions minimize

$$S_0 := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}}.$$

Moreover there holds

$$\int_{\mathbb{R}^N} |\nabla U_{\delta,\zeta}|^2 dx = \int_{\mathbb{R}^N} |U_{\delta,\zeta}|^{2^*} dx = S_0^{\frac{N}{2}}.$$

The second limiting problem is

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.4}$$

For $0 < \mu < \overline{\mu}$ we know from [12, 33] that all positive solutions to (1.4) are given by

$$V_\sigma = C_\mu \left(\frac{\sigma}{\sigma^2 |x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}}$$

with $\sigma > 0$, $\beta_1 := (\sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu})/\sqrt{\overline{\mu}}$, $\beta_2 := (\sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu})/\sqrt{\overline{\mu}}$, and $C_\mu := \left(\frac{4N(\overline{\mu}-\mu)}{N-2} \right)^{\frac{N-2}{4}}$. These solutions are minimizers of

$$S_\mu := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \mu \frac{u^2}{|x|^2}) dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}},$$

and they satisfy

$$\int_{\mathbb{R}^N} \left(|\nabla V_\sigma|^2 - \mu \frac{|V_\sigma|^2}{|x|^2} \right) dx = \int_{\mathbb{R}^N} |V_\sigma|^{2^*} dx = S_\mu^{\frac{N}{2}}.$$

The Green's function of the Dirichlet Laplacian can be written as $G(x,y) = \frac{1}{|x-y|^{N-2}} - H(x,y)$, for $x, y \in \Omega$, where H is the regular part. These functions are symmetric: $G(x,y) = G(y,x)$ and $H(x,y) = H(y,x)$. We need the map

$$\varphi(x) := H^{\frac{1}{2}}(0,0) H^{\frac{1}{2}}(x,x) + G(x,0).$$

If the domain satisfies the symmetry condition

(S_1) Ω is invariant under the reflection $(x_1, x') \mapsto (x_1, -x')$, where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$,

then we define $a < 0 < b$ by $I := \{(t, 0, \dots, 0) : a < t < b\} \subset \Omega$ and $\partial I = \{(a, 0, \dots, 0), (b, 0, \dots, 0)\} \subset \partial\Omega$. For $a \le s, t \le b$ we set

$$g(t,s) := G((t,0,\dots,0),(s,0,\dots,0)), \quad h(t,s) := H((t,0,\dots,0),(s,0,\dots,0)),$$

and

$$\varphi(t) := \varphi(t, 0, \dots, 0) = h^{\frac{1}{2}}(0,0)h^{\frac{1}{2}}(t,t) + g(t,0).$$

Observe that $\varphi(t) \to \infty$ as $t \to a$ or $t \to b$. In particular, φ has a minimum in (a,b).

Now we state the main results of the paper. Throughout the paper let $\Omega \subset \mathbb{R}^N$, $N \geq 7$, be a smooth bounded domain. We begin with the existence of nodal solutions with bubbles concentrating at different points, one being the origin.

Theorem 1.1. *Let $\mu = \mu_0 \varepsilon^\alpha$ with $\mu_0 > 0$ and $\alpha > \frac{N-4}{N-2}$ fixed.*

a) There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exist a pair of solutions $\pm u_\varepsilon$ to problem (1.1) *satisfying*

$$u_\varepsilon(x) = C_\mu \left(\frac{\sigma^\varepsilon}{(\sigma^\varepsilon)^2 |x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}} - C_0 \left(\frac{\delta^\varepsilon}{(\delta^\varepsilon)^2 + |x - \xi^\varepsilon|^2} \right)^{\frac{N-2}{2}} + o(1), \tag{1.5}$$

where $\delta^\varepsilon = \lambda^\varepsilon \varepsilon^{\frac{1}{N-2}}$, $\sigma^\varepsilon = \overline{\lambda}^\varepsilon \varepsilon^{\frac{1}{N-2}}$, and for some $\eta > 0$ small enough, $|\xi^\varepsilon| > \eta$, $\mathrm{dist}(\xi^\varepsilon, \partial\Omega) > \eta$, *$\lambda^\varepsilon$, $\overline{\lambda}^\varepsilon \in (\eta, \frac{1}{\eta})$. Moreover, $\xi^\varepsilon \to \xi^*$ with $\varphi(\xi^*) = \min\limits_{\Omega} \varphi$.*

b) If (S_1) holds there exist solutions as in a) with $\xi^\varepsilon = (t^\varepsilon, 0, \cdots, 0)$ and $t^\varepsilon \to t^*$ as $\varepsilon \to 0$ where t^* is a (local) minimum of $\varphi(t)$ in $(a,b) \setminus \{0\}$.

We can obtain more solutions if $\Omega = B(0,1)$. More precisely, for $k = 2, 3$ we obtain the existence of solutions with $k+1$ bubbles, one positive and k negative.

Theorem 1.2. *Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $\mu = \mu_0 \varepsilon^\alpha$ with $\mu_0 > 0$ and $\alpha > \frac{N-4}{N-2}$ fixed. For $k = 2$ and $N \geq 7$, or $k = 3$ and N large enough, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there exist two pairs of solutions $\pm u_\varepsilon^1$, $\pm u_\varepsilon^2$ of problem* (1.1) *satisfying*

$$u_\varepsilon^j(x) = C_\mu \left(\frac{\sigma_j^\varepsilon}{(\sigma_j^\varepsilon)^2 |x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}} - C_0 \sum_{i=1}^k \left(\frac{\delta_j^\varepsilon}{(\delta_j^\varepsilon)^2 + |x - (\xi_j^\varepsilon)_i|^2} \right)^{\frac{N-2}{2}} + o(1), \tag{1.6}$$

where $\delta_j^\varepsilon = \lambda_j^\varepsilon \varepsilon^{\frac{1}{N-2}}$, $(\xi_j^\varepsilon)_i = (e^{2\pi i \sqrt{-1}/k} \widetilde{\xi}_j^\varepsilon, 0)$, $\widetilde{\xi}_j^\varepsilon \in B^{(2)} := \{x = (x_1, x_2, 0, \cdots, 0) \in B(0,1)\}$, $\sigma_j^\varepsilon = \overline{\lambda}_j^\varepsilon \varepsilon^{\frac{1}{N-2}}$, and for some $\eta > 0$ small enough, $\eta < |\widetilde{\xi}_j^\varepsilon| < 1 - \eta$, λ_j^ε, $\overline{\lambda}_j^\varepsilon \in (\eta, \frac{1}{\eta})$, $i = 1, 2, \dots, k$, $j = 1, 2$.

We would like to point out that the idea of Theorem 1.2 is not applicable for $k = 4$; see Remark 3.6 and Proposition 3.7. It seems very difficult to prove the existence of nodal solutions with the shape as in (1.6) to problem (1.1) for $k \geq 4$. However, for $\Omega = B(0,1)$, the existence of solutions with 5 bubbles, 3 being positive and 2 being negative, can be proved.

Theorem 1.3. *Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $N \geq 7$, $\mu = \mu_0 \varepsilon^\alpha$ with $\mu_0 > 0$ and $\alpha > \frac{N-4}{N-2}$ fixed. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exist one pairs of* 5*-bubble solutions $\pm u_\varepsilon$ to problem* (1.1) *satisfying*

$$u_\varepsilon(x) = C_\mu \left(\frac{\sigma^\varepsilon}{(\sigma^\varepsilon)^2 |x|^{\beta_1} + |x|^{\beta_2}} \right)^{\frac{N-2}{2}} + C_0 \sum_i^4 (-1)^i \left(\frac{\delta^{i,\varepsilon}}{(\delta^{i,\varepsilon})^2 + |x - (\xi^\varepsilon)_i|^2} \right)^{\frac{N-2}{2}} + o(1),$$

where $\delta^{i,\varepsilon} = \lambda^{i,\varepsilon} \varepsilon^{\frac{1}{N-2}}$, $\lambda^{1,\varepsilon} = \lambda^{3,\varepsilon}$, $\lambda^{2,\varepsilon} = \lambda^{4,\varepsilon}$, $(\xi^\varepsilon)_i = (e^{2\pi i \sqrt{-1}/k} \widetilde{\xi}^\varepsilon, 0)$, $\widetilde{\xi}^\varepsilon \in B^{(2)}$, $\sigma^\varepsilon = \overline{\lambda}^\varepsilon \varepsilon^{\frac{1}{N-2}}$, and for some $\eta > 0$ small enough, $\eta < |\widetilde{\xi}^\varepsilon| < 1 - \eta$, $\lambda^{i,\varepsilon}, \overline{\lambda}^\varepsilon \in (\eta, \frac{1}{\eta})$, $i = 1, \dots, 4$.

The assumption that $\alpha > \frac{N-4}{N-2}$ is critical to obtain the existence of nodal solutions with multiple bubbles concentrating at different points since it can be seen from the reduction procedure in Section 3 that the reduced function has no critical point if $0 \le \alpha \le \frac{N-4}{N-2}$. It is natural to ask, whether there exists $\widetilde{\mu}(\varepsilon) > 0$ for $\varepsilon > 0$ small, such that problem (1.1) admits positive or nodal solutions with multiple bubbles concentrating at different points when $0 \le \mu \le \widetilde{\mu}(\varepsilon)$.

Now we state a result about the existence of nodal solutions that are towers of bubbles concentrating at the origin.

Theorem 1.4. *Let $\mu = \mu_0\varepsilon$ with $\mu_0 > 0$ fixed. For any given integer $k \ge 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exist a pair of solutions $\pm u_\varepsilon$ to problem* (1.1) *satisfying that, as $\varepsilon \to 0^+$,*

$$u_\varepsilon(x) = C_\mu(-1)^k \left(\frac{\sigma^\varepsilon}{(\sigma^\varepsilon)^2|x|^{\beta_1} + |x|^{\beta_2}}\right)^{\frac{N-2}{2}} + C_0 \sum_{i=1}^{k}(-1)^{i-1}\left(\frac{\delta_i^\varepsilon}{(\delta_i^\varepsilon)^2 + |x-\xi_i^\varepsilon|^2}\right)^{\frac{N-2}{2}} + o(1)$$

where $\delta_i^\varepsilon = \lambda_i^\varepsilon \varepsilon^{\frac{2i-1}{N-2}}$, $\xi_i^\varepsilon = \delta_i^\varepsilon \zeta_i^\varepsilon$, $\zeta_i^\varepsilon \in \mathbb{R}^N$, $i = 1, 2, \dots, k$, $\sigma^\varepsilon = \overline{\lambda}^\varepsilon \varepsilon^{\frac{2(k+1)-1}{N-2}}$, and for some $\eta > 0$ small enough, $\lambda_i^\varepsilon, \overline{\lambda}^\varepsilon \in \left(\eta, \frac{1}{\eta}\right)$ and $|\zeta_i^\varepsilon| \le \frac{1}{\eta}$ for $i = 1, \dots, k$.

We assume $N \ge 7$ in this paper for technical reasons and in order to not to make the presentation too heavy. The results can be extended to the case $N = 6$. For $N \le 5$, there would be technical difficulties.

The paper is organized as follows. In Section 2, we give some notations and preliminary results. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2, that is, the existence of nodal solutions with multiple bubbles blowing up at different points. The proof of Theorem 1.4, the existence of nodal bubble tower solutions, is given in Section 4. At last, some useful technical lemmas are collected in the appendices.

2 Notations and preliminary results

Throughout this paper, positive constants will be denoted by C, c.

As in [15] let $\iota^* : L^{2N/(N+2)}(\Omega) \to H_\mu(\Omega)$ be the adjoint operator of the inclusion $\iota : H_\mu(\Omega) \to L^{2N/(N-2)}(\Omega)$, that is,

$$\iota^*(u) = v \qquad \Longleftrightarrow \qquad (v, \phi) = \int_\Omega u(x)\phi(x)dx, \quad \text{for all } \phi \in H_\mu(\Omega). \tag{2.1}$$

This is continuous, i.e., there exists $c > 0$ such that

$$\|\iota^*(u)\|_\mu \le c\|u\|_{2N/(N+2)}. \tag{2.2}$$

Then problem (1.1) is equivalent to the fixed point problem

$$u = \iota^*(f_\varepsilon(u)), u \in H_\mu(\Omega),$$

where $f_\varepsilon(s) = |s|^{2^*-2-\varepsilon}s$.

To continue, we show an eigenvalue problem first.

Proposition 2.1. *Let Λ_i, $i = 1, 2, \dots,$ be the eigenvalues of*

$$\begin{cases} -\Delta u - \mu\frac{u}{|x|^2} = \Lambda|V_\sigma|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ |u| \to 0 & \text{as } |x| \to +\infty \end{cases} \tag{2.3}$$

in increasing order. Then $\Lambda_1 = 1$ *with eigenfunction* V_σ*,* $\Lambda_2 = 2^* - 1$ *with eigenfunction* $\frac{\partial V_\sigma}{\partial \sigma}$*.*

Proof. Direct computations give that V_σ and $\frac{\partial V_\sigma}{\partial \sigma}$ are eigenfunctions corresponding to 1 and $2^* - 1$, respectively. Now as in [34], it is enough to prove that the eigenfunction u corresponding to the eigenvalue $\Lambda \le 2^* - 1$ has to be radial.

Denote by ψ_i, $i = 0, 1, 2, \ldots$, the sequence of spherical harmonics, which are eigenfunctions of the Laplace-Beltrami operator on S^{N-1}:

$$-\Delta_{S^{N-1}} \psi_i = \tau_i \psi_i.$$

It is well known that $\tau_0 = 0$, $\tau_1, \ldots, \tau_N = N - 1$, $\tau_{N+1} > \tau_N$. We prove that for every $i \ge 1$,

$$\int_{S^{N-1}} u(r,\theta)\psi_i(\theta) d\theta = 0.$$

Setting $\varphi_i(r) = \int_{S^{N-1}} u(r,\theta)\psi_i(\theta) d\theta$ we have:

$$\begin{aligned}
\Delta \varphi_i &= \Delta_r \varphi_i = \int_{S^{N-1}} \Delta_r u(r,\theta)\psi_i(\theta) d\theta \\
&= -\int_{S^{N-1}} \frac{\Delta_\theta u(r,\theta)}{r^2} \psi_i(\theta) d\theta - \int_{S^{N-1}} \left(\frac{\mu u(r,\theta)}{r^2} + \Lambda V_\sigma^{2^*-2} u(r,\theta) \right) \psi_i(\theta) d\theta \\
&= \int_{S^{N-1}} \frac{\tau_i u(r,\theta)}{r^2} \psi_i(\theta) d\theta - \int_{S^{N-1}} \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} \right) u(r,\theta)\psi_i(\theta) d\theta \\
&= \left(\frac{\tau_i}{r^2} - \left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2}\right) \right) \varphi_i(r).
\end{aligned}$$

This implies for any $R > 0$:

$$\begin{aligned}
0 &= \int_{B_R(0)} \Delta \varphi_i \frac{\partial V_\sigma}{\partial r} + \left(\left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} \right) - \frac{\tau_i}{r^2} \right) \varphi_i \frac{\partial V_\sigma}{\partial r} \\
&= \int_{B_R(0)} \varphi_i \Delta \left(\frac{\partial V_\sigma}{\partial r} \right) + \left(\left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} \right) - \frac{\tau_i}{r^2} \right) \varphi_i \frac{\partial V_\sigma}{\partial r} + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r} \cdot \frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2} \right) \\
&= \int_{B_R(0)} \frac{N-1}{r^2} \varphi_i \frac{\partial V_\sigma}{\partial r} + \varphi_i \frac{\partial}{\partial r} \left(-\mu \frac{V_\sigma}{r^2} - V_\sigma^{2^*-1} \right) + \left(\left(\frac{\mu}{r^2} + \Lambda V_\sigma^{2^*-2} \right) - \frac{\tau_i}{r^2} \right) \varphi_i \frac{\partial V_\sigma}{\partial r} \\
&\quad + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r} \cdot \frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2} \right) \\
&= \int_{B_R(0)} \frac{N-1-\tau_i}{r^2} \varphi_i \frac{\partial V_\sigma}{\partial r} + (\Lambda - (2^*-1)) V_\sigma^{2^*-2} \frac{\partial V_\sigma}{\partial r} \varphi_i + \frac{2\mu V_\sigma}{r^3} \varphi_i \\
&\quad + \int_{\partial B_R(0)} \left(\frac{\partial V_\sigma}{\partial r} \cdot \frac{\partial \varphi_i}{\partial r} - \varphi_i \frac{\partial^2 V_\sigma}{\partial r^2} \right).
\end{aligned}$$

Now let R be the first zero of φ_i; $R := +\infty$ if φ_i is never zero. Without loss of generality we assume $\varphi_i(r) > 0$ for $r \in (0, R)$. Then $\frac{\partial \varphi_i}{\partial r}(R) \le 0$, and we finish the proof. □

Let us define the projection $P : H^1(\mathbb{R}^N) \to H_0^1(\Omega)$, that is, $\Delta P u = \Delta u$ in Ω, $Pu = 0$ on $\partial\Omega$.

Proposition 2.2. *Let* $0 \in \Omega$ *be a smooth bounded domain. Denote* $\varphi_\sigma := V_\sigma - PV_\sigma$*. Then*

$$0 \le \varphi_\sigma \le V_\sigma, \quad \textit{where } \varphi_\sigma(x) = C_\mu \overline{d}^{\sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}}(x) H(0,x) \sigma^{\frac{N-2}{2}} + \hbar_\sigma; \tag{2.4}$$

here $d_{\inf} \le \overline{d} \le d_{\sup}$*,* $d_{\inf} = \operatorname{dist}(0, \partial\Omega) = \inf\{|x| : x \in \partial\Omega\}$*,* $d_{\sup} = \sup\{|x| : x \in \partial\Omega\}$*, and* $\hbar_\sigma$ *satisfies the uniform estimates*

$$\hbar_\sigma = O(\sigma^{\frac{N+2}{2}}), \quad \frac{\partial \hbar_\sigma}{\partial \sigma} = O(\sigma^{\frac{N}{2}}). \tag{2.5}$$

Proof. It is easy to see that φ_σ satisfies

$$\begin{cases}\Delta\varphi_\sigma(x)=0 & \text{in } \Omega\backslash\{0\},\\ \varphi_\sigma(x)=V_\sigma(x)=C_\mu(\frac{\sigma}{\sigma^2|x|^{\beta_1}+|x|^{\beta_2}})^{\frac{N-2}{2}} & \text{on } \partial\Omega.\end{cases}$$

Then the first part of (2.4) holds by the maximum principle.

Consider the function H satisfying

$$\begin{cases}\Delta H(0,x)=0 & \text{in } \Omega\backslash\{0\},\\ H(0,x)=\frac{1}{|x|^{N-2}} & \text{on } \partial\Omega.\end{cases}$$

Notice that on $\partial\Omega$,

$$\varphi_\sigma - C_\mu d_{\text{inf}}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}H(0,x)\sigma^{\frac{N-2}{2}} = C_\mu\sigma^{\frac{N-2}{2}}[\frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}-\frac{d_{\text{inf}}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}}{|x|^{N-2}}]\geq O(\sigma^{\frac{N+2}{2}}),$$

and

$$\varphi_\sigma - C_\mu d_{\text{sup}}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}H(0,x)\sigma^{\frac{N-2}{2}} = C_\mu\sigma^{\frac{N-2}{2}}[\frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}-\frac{d_{\text{sup}}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}}{|x|^{N-2}}]\leq O(\sigma^{\frac{N+2}{2}}).$$

Then the maximum principle and direct computations yield the second part of (2.4) and (2.5). □

Remark 2.3. a) If $\mu\to 0^+$, then

$$\varphi_\sigma(x)=C_0H(0,x)\sigma^{\frac{N-2}{2}}+O(\mu\sigma^{\frac{N-2}{2}})+\hbar_\sigma. \tag{2.6}$$

b) Let us recall the similar results for $U_{\delta,\xi}$ obtained in [28], that is

$$0\leq\varphi_{\delta,\xi}:=U_{\delta,\xi}-PU_{\delta,\xi}\leq U_{\delta,\xi},\ \varphi_{\delta,\xi}=C_0H(\xi,\cdot)\delta^{\frac{N-2}{2}}+\hbar_{\delta,\xi}, \tag{2.7}$$

where $\hbar_{\delta,\xi}=O(\delta^{\frac{N+2}{2}})$.

3 Solutions with multiple bubbles concentrating at different points

3.1 The finite dimensional reduction

We introduce some notation. Fix an integer $k\geq 0$. For $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_k,\overline{\lambda})\in\mathbb{R}_+^{k+1}$ and $\xi=(\xi_1,\xi_2,\dots,\xi_k)\in\Omega^k$ we define

$$W_{\varepsilon,\lambda,\xi}:=\sum_{i=1}^k \text{Ker}\left(-\Delta-(2^*-1)U_{\delta_i,\xi_i}^{2^*-2}\right)+\text{Ker}\left(-\Delta-\frac{\mu}{|x|^2}-(2^*-1)V_\sigma^{2^*-2}\right),$$

where $\delta_i=\lambda_i\varepsilon^{\frac{1}{N-2}}$, $\sigma=\overline{\lambda}\varepsilon^{\frac{1}{N-2}}$. From [7], the kernel of the operator $-\Delta-(2^*-1)U_{\delta_i,\xi_i}^{2^*-2}$ on $L^2(\mathbb{R}^N)$ has dimension $N+1$ and is spanned by $\frac{\partial U_{\delta_i,\xi_i}}{\partial\delta_i}$, $\frac{\partial U_{\delta_i,\xi_i}}{\partial(\xi_i)^j}$, $j=1,2,\dots,N$, where $(\xi_i)^j$ is the $j-$th component of ξ_i. Combining this with Proposition 2.1, we have

$$W_{\varepsilon,\lambda,\xi}=\text{span}\left\{\Psi_i^j,\ \Psi_i^0,\ \overline{\Psi},\ i=1,2,\dots,k,\ j=1,2,\dots,N\right\},$$

where for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N$:

$$\Psi_i^j := \frac{\partial U_{\delta_i,\xi_i}}{\partial (\xi_i)^j}, \quad \Psi_i^0 := \frac{\partial U_{\delta_i,\xi_i}}{\partial \delta_i}, \quad \overline{\Psi} := \frac{\partial V_\sigma}{\partial \sigma}.$$

For $\eta \in (0,1)$ we define

$$\mathcal{O}_\eta := \big\{(\lambda,\xi) \in \mathbb{R}_+^{k+1} \times \Omega^k : \lambda_i \in (\eta, \eta^{-1}), \overline{\lambda} \in (\eta, \eta^{-1}),\ \mathrm{dist}(\xi_i, \partial\Omega) > \eta,$$
$$|\xi_i| > \eta,\ |\xi_{i_1} - \xi_{i_2}| > \eta,\ i, i_1, i_2 = 1, 2, \ldots, k,\ i_1 \neq i_2\big\}.$$

Let us introduce the spaces

$$K_{\varepsilon,\lambda,\xi} := PW_{\varepsilon,\lambda,\xi},$$

and

$$K_{\varepsilon,\lambda,\xi}^\perp := \{\phi \in H_\mu(\Omega) : (\phi, P\Psi) = 0, \text{ for all } \Psi \in W_{\varepsilon,\lambda,\xi}\},$$

and the $(\cdot,\cdot)$-orthogonal projections

$$\Pi_{\varepsilon,\lambda,\xi} := H_\mu(\Omega) \to K_{\varepsilon,\lambda,\xi},$$

and

$$\Pi_{\varepsilon,\lambda,\xi}^\perp := Id - \Pi_{\varepsilon,\lambda,\xi} : H_\mu(\Omega) \to K_{\varepsilon,\lambda,\xi}^\perp.$$

Solving problem (1.1) is equivalent to finding $\eta > 0$, $\varepsilon > 0$, $(\lambda,\xi) \in \mathcal{O}_\eta$ and $\phi_{\varepsilon,\lambda,\xi} \in K_{\varepsilon,\lambda,\xi}^\perp$ such that:

$$\Pi_{\varepsilon,\lambda,\xi}^\perp \left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}))\right) = 0, \tag{3.1}$$

and

$$\Pi_{\varepsilon,\lambda,\xi} \left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}))\right) = 0, \tag{3.2}$$

where

$$V_{\varepsilon,\lambda,\xi} = -\sum_{i=1}^k PU_{\delta_i,\xi_i} + PV_\sigma \tag{3.3}$$

or

$$V_{\varepsilon,\lambda,\xi} = \sum_{i=1}^k (-1)^i PU_{\delta_i,\xi_i} + PV_\sigma. \tag{3.4}$$

In the rest of this section, we only consider $V_{\varepsilon,\lambda,\xi}$ as in (3.3) because the argument for the solutions of the form (3.4) is similar.

We prove (3.1) first. Let us introduce the operator $L_{\varepsilon,\lambda,\xi} : K_{\varepsilon,\lambda,\xi}^\perp \to K_{\varepsilon,\lambda,\xi}^\perp$ defined by

$$L_{\varepsilon,\lambda,\xi}(\phi) = \phi - \Pi_{\varepsilon,\lambda,\xi}^\perp \iota^*(f_0'(V_{\varepsilon,\lambda,\xi})\phi).$$

Proposition 3.1. *For any $\eta > 0$, there exist $\varepsilon_0 > 0$ and $c > 0$ such that for every $(\lambda,\xi) \in \mathcal{O}_\eta$ and for every $\varepsilon \in (0, \varepsilon_0)$:*

$$\|L_{\varepsilon,\lambda,\xi}(\phi)\|_\mu \geq c\|\phi\|_\mu, \quad \text{for all } \phi \in K_{\varepsilon,\lambda,\xi}^\perp.$$

In particular, $L_{\varepsilon,\lambda,\xi}$ is invertible with continuous inverse.

Proof. We argue by contradiction, following the same line as in [24]. Suppose there exist $\eta > 0$, sequences $\varepsilon^n > 0$, $(\lambda^n, \xi^n) \in \mathcal{O}_\eta$, $\phi^n \in H_\mu(\Omega)$ such that $\varepsilon^n \to 0$, $\lambda^n = (\lambda_1^n, \ldots, \lambda_k^n, \overline{\lambda}^n) \to (\lambda_1, \ldots \lambda_k, \overline{\lambda})$, $\xi^n = (\xi_1^n, \ldots, \xi_k^n) \to (\xi_1, \ldots \xi_k)$, as $n \to \infty$, and

$$\phi^n \in K^{\perp}_{\varepsilon^n,\lambda^n,\xi^n}, \|\phi^n\|_\mu = 1, \tag{3.5}$$

$$L_{\varepsilon^n,\lambda^n,\xi^n}(\phi^n) = h^n, \text{with } \|h^n\|_\mu \to 0. \tag{3.6}$$

Thus we have

$$\phi^n - \iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n) = h^n - \Pi_{\varepsilon^n,\lambda^n,\xi^n}(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)). \tag{3.7}$$

Setting

$$\delta_i^n = \lambda_i^n \varepsilon^{\frac{1}{N-2}}, \quad \sigma^n = \overline{\lambda}^n \varepsilon^{\frac{1}{N-2}}$$

and

$$(\Psi_i^j)_n := \frac{\partial U_{\delta_i^n,\xi_i^n}}{\partial (\xi_i^n)^j} \text{ for } j = 1,2,\ldots,N, \quad (\Psi_i^0)_n := \frac{\partial U_{\delta_i^n,\xi_i^n}}{\partial \delta_i^n}, \quad (\overline{\Psi})_n := \frac{\partial V_{\sigma^n}}{\partial \sigma^n},$$

where $(\xi_i^n)^j$ is the $j-$th component of ξ_i^n, we obtain

$$w^n := -\Pi_{\varepsilon^n,\lambda^n,\xi^n}(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)) = \sum_{i=1}^k \sum_{j=0}^N c_{i,j}^n P(\Psi_i^j)_n + c_0^n P(\overline{\Psi})_n$$

for some coefficients $c_{i,j}^n, c_0^n$. Now we argue in three steps.

Step 1. We prove

$$\lim_{n\to\infty} \|w^n\|_\mu = 0. \tag{3.8}$$

Multiplying (3.7) by $\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2}$, we get

$$\begin{aligned}
&\int_\Omega \phi^n \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2}\right) - \int_\Omega \iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n) \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2}\right) \\
&\quad = -\int_\Omega h^n \left(-\Delta P(\Psi_l^h)_n - \mu \frac{P(\Psi_l^h)_n}{|x|^2}\right) + \int_\Omega w^n \left(\Delta P(\Psi_l^h)_n + \mu \frac{P(\Psi_l^h)_n}{|x|^2}\right)
\end{aligned}$$

and then

$$\begin{aligned}
&\sum_{i=1}^k \sum_{j=0}^N c_{i,j}^n \big(P(\Psi_i^j)_n, P(\Psi_l^h)_n\big) + c_0^n \big(P(\overline{\Psi})_n, P(\Psi_l^h)_n\big) \\
&\qquad = \big(\phi^n, P(\Psi_l^h)_n\big) - \big(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n), P(\Psi_l^h)_n\big) - \big(h^n, P(\Psi_l^h)_n\big).
\end{aligned}$$

From Lemma A.1 we deduce:

$$c_{l,h}^n \tilde{c}_{l,h}^n \frac{1}{(\delta_l^n)^2} + o\left(\frac{1}{(\delta_l^n)^2}\right) = -\big(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n), P(\Psi_l^h)_n\big), \tag{3.9}$$

where $\tilde{c}_{l,h}^n > 0$ is a constant.

Proposition 2.2 implies

$$\begin{aligned}
0 &= (\phi^n, P(\Psi_l^h)_n) = \int_\Omega \nabla\phi^n \nabla P(\Psi_l^h)_n - \mu \frac{\phi^n P(\Psi_l^h)_n}{|x|^2} \\
&= \int_\Omega \nabla\phi^n \nabla(\Psi_l^h)_n - \mu \frac{\phi^n (\Psi_l^h)_n}{|x|^2} + o(1) \\
&= \int_\Omega f_0'(U_{\delta_l^n,\xi_l^n})(\Psi_l^h)_n \phi^n + o(1),
\end{aligned}$$

and then

$$
\begin{aligned}
&-(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n),P(\Psi_l^h)_n)=-\int_\Omega f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n P(\Psi_l^h)_n\\
&\quad\le\left|\int_\Omega\big(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})-f_0'(U_{\delta_l^n,\xi_l^n})\big)\phi^n(\Psi_l^h)_n\right|+\left|\int_\Omega f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n\big(P(\Psi_l^h)_n-(\Psi_l^h)_n\big)\right|+o(1)\\
&\quad=o(1)
\end{aligned}
$$

by Lemma A.2 and Lemma A.3.

Combining the above inequality with (3.9) yields $c_{l,h}^n\to 0$ as $n\to\infty$. Similar arguments show that $c_0^n\to 0$ as $n\to\infty$, and $\lim\limits_{n\to\infty}\|w^n\|_\mu=0$ follows.

Step 2. Let $\chi:\mathbb{R}^N\to[0,1]$ be a smooth cut-off function, such that $\chi(x)=1$ if $|x|\le\eta/4$, $\chi(x)=0$ if $|x|\ge\eta/2$, and $|\nabla\chi(x)|\le\frac{C}{\eta}$. We set

$$
\phi_i^n(x):=\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\chi\big((\varepsilon^n)^{\alpha_1}x\big),\quad x\in\Omega_i^n:=\frac{\Omega-\xi_i^n}{(\varepsilon^n)^{\alpha_1}},\ i=1,\dots,k,
$$

and

$$
\phi_0^n(x):=\big((\varepsilon^n)^{\alpha_2}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_2}x\big)\chi\big((\varepsilon^n)^{\alpha_2}x\big),\quad x\in\Omega_0^n:=\frac{\Omega}{(\varepsilon^n)^{\alpha_2}},
$$

where α_1,α_2 are positive constants which will be determined later. Since ϕ_i^n is bounded in $D^{1,2}(\mathbb{R}^N)$, we may assume, up to a subsequence,

$$
\phi_i^n\rightharpoonup\phi_i^\infty\quad\text{weakly in } D^{1,2}(\mathbb{R}^N),\ i=0,1,2,\dots,k.
$$

Now we claim that

$$
\phi_i^\infty(x)=0,\quad i=0,1,\dots,k. \tag{3.10}
$$

Firstly we prove (3.10) for $i=1,\dots,k$. Notice that $\big|\nabla\chi\big((\varepsilon^n)^{\alpha_1}x\big)\big|=\big|(\varepsilon^n)^{\alpha_1}\nabla\chi(\cdot)\big|\le\frac{C(\varepsilon^n)^{\alpha_1}}{\eta}=o(1)$. Thus we have for any $\psi\in C_0^\infty(\mathbb{R}^N)$:

$$
\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\int_{\Omega_i^n}\nabla\chi\big((\varepsilon^n)^{\alpha_1}x\big)\big(\phi^n((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\nabla\psi-\psi\nabla\phi^n((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\big)=o(1). \tag{3.11}
$$

On the other hand, taking $\alpha_1=\frac{1}{N-2}$ and noticing $N\ge 7$, we get:

$$
\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{2-N}{2}}\mu\int_\Omega\frac{\iota^*\big(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n}(y))\phi^n(y)\big)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right)}{|y|^2}=o(1). \tag{3.12}
$$

By (3.11), (3.7), (3.6), (3.8), (3.12) and (2.7), we have for any $\psi \in C_0^\infty(\mathbb{R}^N)$:

$$
\begin{aligned}
&\int_{\Omega_i^n} \nabla\phi_i^n \nabla\psi \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \Big(\nabla\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\nabla(\chi((\varepsilon^n)^{\alpha_1}x)\psi) \\
&\qquad + \nabla\chi\big((\varepsilon^n)^{\alpha_1}x\big)\big(\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\nabla\psi - \psi\nabla\phi^n((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\big)\Big) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \nabla\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\nabla\big(\chi((\varepsilon^n)^{\alpha_1}x)\psi\big) + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \nabla\iota^*\big(f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\big)\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\big)\nabla\big(\chi((\varepsilon^n)^{\alpha_1}x)\psi\big) \\
&\qquad + \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \nabla h_n((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\nabla\big(\chi((\varepsilon^n)^{\alpha_1}x)\psi\big) \\
&\qquad + \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \nabla w_n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\nabla(\chi((\varepsilon^n)^{\alpha_1}x)\psi) + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}} \int_{\Omega_i^n} \nabla\iota^*\big(f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}((\varepsilon^n)^{\alpha_1}x+\xi_i^n)\big)\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\big)\nabla\big(\chi((\varepsilon^n)^{\alpha_1}x)\psi\big) \\
&\qquad + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{2-N}{2}} \int_{\Omega} \nabla\iota^*\big(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n}(y))\phi^n(y)\big)\nabla\big(\chi(y-\xi_i^n)\psi(\frac{y-\xi_i^n}{(\varepsilon^n)^{\alpha_1}})\big) + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{2-N}{2}} \int_{\Omega} f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}(y)\big)\phi^n(y)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right) + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{2-N}{2}} \int_{|y-\xi_i^n|\le\eta/2} f_0'\left(-\sum_{j=1}^{k} PU_{\delta_j^n,\xi_j^n}(y) + PV_{\sigma^n}(y)\right)\phi^n(y)\chi(y-\xi_i^n)\psi\left(\frac{y-\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right) \\
&\qquad + o(1) \\
&= \big((\varepsilon^n)^{\alpha_1}\big)^{\frac{2-N}{2}} \int_{|y-\xi_i^n|\le\eta/2} f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)\phi^n(y)\chi(y-\xi_i^n)\psi(\frac{y-\xi_i^n}{(\varepsilon^n)^{\alpha_1}}) + o(1) \\
&= \int_{|(\varepsilon^n)^{\alpha_1}x|\le\eta/2} f_0'\big(U_{\lambda_i^n,0}(x)\big)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\chi\big((\varepsilon^n)^{\alpha_1}x\big)\psi(x) + o(1) \\
&= \int_{\mathbb{R}^N} f_0'(U_{\lambda_i,0}(x))\phi_i^\infty(x)\psi(x) + o(1).
\end{aligned} \tag{3.13}
$$

Therefore ϕ_i^∞ is a weak solution of

$$
-\Delta\phi_i^\infty = f_0'(U_{\lambda_i,0})\phi_i^\infty \quad \text{in } D^{1,2}(\mathbb{R}^N). \tag{3.14}
$$

In order to continue we denote $\Psi^j_{\lambda_i,0} := \frac{\partial U_{\lambda_i,0}}{\partial x^j}$ for $j = 1,\dots,N$, and $\Psi^0_{\lambda_i,0} := \frac{\partial U_{\lambda_i,0}}{\partial \lambda_i}$. Now we claim that

$$
\int_{\mathbb{R}^N} \nabla\phi_i^\infty(x)\nabla\Psi^j_{\lambda_i,0}(x) = 0,\ j = 0,1,\dots,N. \tag{3.15}
$$

In fact,

$$
\begin{aligned}
&\left|\int_{\Omega_i^n} f_0'\big(U_{\lambda_i^n,0}(x)\big)\phi_i^n(x)\Psi^j_{\lambda_i^n,0}(x)\right| \qquad (3.16)\\
&=\left|\int_{\Omega_i^n} f_0'\big(U_{\lambda_i^n,0}(x)\big)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x+\xi_i^n\big)\chi((\varepsilon^n)^{\alpha_1}x)\Psi^j_{\lambda_i^n,0}(x)\right|\\
&=\left|\int_{(\varepsilon^n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}}(x)\right)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x\big)\chi\big((\varepsilon^n)^{\alpha_1}x-\xi_i^n\big)\Psi^j_{\lambda_i^n,0}\big(x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\big)\right|.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\int_{(\varepsilon^n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}}(x)\right)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x\big)\Psi^j_{\lambda_i^n,0}\big(x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\big)\\
&\qquad=-(\varepsilon^n)^{\alpha_1}\int_\Omega f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)\phi^n(y)\big(\Psi_i^j\big)_n(y)=o(1),
\end{aligned}
$$

then

$$
\begin{aligned}
&(3.16)\\
&=\left|\int_{(\varepsilon^n)^{-\alpha_1}\Omega} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}}(x)\right)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x\big)\big(\chi((\varepsilon^n)^{\alpha_1}x-\xi_i^n)-1\big)\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right)\right|\\
&\qquad+o(1)\\
&\le\left|\int_{\left|x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon^n)^{\alpha_1}}} f_0'\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}}(x)\right)\big((\varepsilon^n)^{\alpha_1}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_1}x\big)\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right)\right|\\
&\qquad+o(1)\\
&\le C\|\phi^n\|_{\frac{2N}{N-2}}\left(\int_{\left|x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon^n)^{\alpha_1}}}\left(U_{\lambda_i^n,\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}}(x)\right)^{\frac{2N}{N-2}}\right)^{\frac{2}{N}}\\
&\qquad\times\left(\int_{\left|x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right|\ge\frac{\eta/4}{(\varepsilon^n)^{\alpha_1}}}\left(\Psi^j_{\lambda_i^n,0}\left(x-\frac{\xi_i^n}{(\varepsilon^n)^{\alpha_1}}\right)\right)^{\frac{2N}{N-2}}\right)^{\frac{N-2}{2N}}\\
&=o(1).
\end{aligned}
$$

Therefore (3.15) holds. Using this and (3.14) we conclude that (3.10) holds for $i=1,\dots,k$.

Now we turn to the proof of $\phi_0^\infty=0$. Setting $\alpha_2=\frac{1}{N-2}$ we obtain as in (3.13):

$$
\begin{aligned}
\int_{\Omega_0^n}\nabla\phi_0^n\nabla\psi&=\big((\varepsilon^n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_\Omega f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}(y)\big)\phi^n(y)\chi(y)\psi\big((\varepsilon^n)^{-\alpha_2}y\big)+o(1)\\
&=\big((\varepsilon^n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_{|y|\le\eta/2} f_0'\left(-\sum_{j=1}^k PU_{\delta_j^n,\xi_j^n}(y)+PV_{\sigma^n}(y)\right)\phi^n(y)\chi(y)\psi((\varepsilon^n)^{-\alpha_2}y)+o(1)\\
&=\big((\varepsilon^n)^{\alpha_2}\big)^{\frac{2-N}{2}}\int_{|y|\le\eta/2} f_0'\big(V_{\sigma^n}(y)\big)\phi^n(y)\chi(y)\psi((\varepsilon^n)^{-\alpha_2}y)+o(1)\\
&=\int_{|(\varepsilon^n)^{\alpha_2}x|\le\eta/2} f_0'\big(U_{\overline{\lambda}^n,0}(x)\big)\big((\varepsilon^n)^{\alpha_2}\big)^{\frac{N-2}{2}}\phi^n\big((\varepsilon^n)^{\alpha_2}x\big)\chi\big((\varepsilon^n)^{\alpha_2}x\big)\psi(x)+o(1)\\
&=\int_{\mathbb{R}^N} f_0'\big(U_{\overline{\lambda},0}\big)\phi_0^\infty\psi(x)+o(1).
\end{aligned}
$$

Therefore ϕ_0^∞ is a weak solution of

$$-\Delta\phi_0^\infty = f_0'(U_{\overline{\lambda},0})\phi_0^\infty, \quad \text{in } D^{1,2}(\mathbb{R}^N). \tag{3.17}$$

Similarly to (3.15) there holds

$$\int_{\mathbb{R}^N} \nabla\phi_0^\infty(x)\nabla\Psi^j_{\overline{\lambda},0}(x) = 0, \quad \text{for } j = 0, 1, \ldots, N, \tag{3.18}$$

where $\Psi^j_{\overline{\lambda},0} := \frac{\partial U_{\overline{\lambda},0}}{\partial x^j}$, for $j = 1, \ldots, N$, and $\Psi^0_{\overline{\lambda},0} := \frac{\partial U_{\overline{\lambda},0}}{\partial \overline{\lambda}}$. This shows that $\phi_0^\infty = 0$ as claimed.

Step 3. We obtain a contradiction.

First we claim that

$$\lim_{n\to\infty} \int_\Omega f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}(y)\big)(\phi^n(y))^2 = 0. \tag{3.19}$$

In fact, (2.6) and (2.7) imply:

$$\int_\Omega f_0'(V_{\varepsilon^n,\lambda^n,\xi^n}(y))(\phi^n(y))^2 = \int_{B(0,\frac{\eta}{4})\cup\bigcup\limits_{i=1}^k B(\xi_i,\frac{\eta}{4})} f_0'\left(-\sum_{j=1}^k U_{\delta_j^n,\xi_j^n}(y) + V_{\sigma^n}(y)\right)(\phi^n(y))^2 + o(1).$$

Notice that $f_0'(U_{\lambda_i^n,0}) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and (3.10) imply

$$\begin{aligned}
\int_{B(\xi_i,\frac{\eta}{4})} f_0'\left(-\sum_{j=1}^k U_{\delta_j^n,\xi_j^n}(y) + V_{\sigma^n}(y)\right)\big(\phi^n(y)\big)^2 &= \int_{B(\xi_i,\frac{\eta}{4})} f_0'\big(U_{\delta_i^n,\xi_i^n}(y)\big)(\phi^n(y))^2 + o(1) \\
&= \int_{|(\varepsilon^n)^{\alpha_1}x|\le\frac{\eta}{4}} f_0'\big(U_{\lambda_i^n,0}(x)\big)(\phi_i^n(x))^2 + o(1) \\
&= o(1).
\end{aligned} \tag{3.20}$$

Similarly we obtain:

$$\int_{B(0,\frac{\eta}{4})} f_0'\left(-\sum_{j=1}^k U_{\delta_j^n,\xi_j^n}(y) + V_{\sigma^n}(y)\right)(\phi^n(y))^2 = o(1). \tag{3.21}$$

Now we obtain (3.19) from (3.20) and (3.21).

On the other hand, (3.7), (3.6), and (3.8) imply:

$$\begin{aligned}
\int_\Omega |\nabla\phi^n|^2 &= \int_\Omega \nabla\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)\nabla\phi^n + \int_\Omega \nabla h^n\nabla\phi^n + \int_\Omega \nabla w^n\nabla\phi^n \\
&= \int_\Omega \nabla\iota^*\big(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n\big)\nabla\phi^n - \mu\int_\Omega \frac{\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)\phi^n}{|x|^2} + o(1) \\
&= \int_\Omega f_0'\big(V_{\varepsilon^n,\lambda^n,\xi^n}(y)\big)(\phi^n(y))^2 + o(1),
\end{aligned}$$

which contradicts (3.19) using (3.5). □

Proposition 3.2. *For every $\eta > 0$ there exist $\varepsilon_0 > 0$ and $c_0 > 0$ with the following property: for every $(\lambda,\xi) \in \mathcal{O}_\eta$ and for every $\varepsilon \in (0,\varepsilon_0)$ there exists a unique solution $\phi_{\varepsilon,\lambda,\xi} \in K^\perp_{\varepsilon,\lambda,\xi}$ of equation (3.1) satisfying*

$$\|\phi_{\varepsilon,\lambda,\xi}\|_\mu \le c_0\left(\varepsilon^{\frac{N+2}{2(N-2)}} + \varepsilon^{\frac{1+2\alpha}{4}}\right), \tag{3.22}$$

and $\Phi_\varepsilon : \mathcal{O}_\eta \to K^\perp_{\varepsilon,\lambda,\xi}$ defined by $\Phi_\varepsilon(\lambda,\xi) := \phi_{\varepsilon,\lambda,\xi}$ is C^1.

Proof. As in [4] solving (3.1) is equivalent to finding a fixed point of the operator $T_{\varepsilon,\lambda,\xi} : K^{\perp}_{\varepsilon,\lambda,\xi} \to K^{\perp}_{\varepsilon,\lambda,\xi}$ defined by

$$T_{\varepsilon,\lambda,\xi}(\phi) = L^{-1}_{\varepsilon,\lambda,\xi}\Pi^{\perp}_{\varepsilon,\lambda,\xi}\big(\iota^*\big(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi\big) - V_{\varepsilon,\lambda,\xi}\big).$$

We claim that $T_{\varepsilon,\lambda,\xi}$ is a contraction mapping.

First of all, Proposition 3.1, Lemma A.4 and (2.2) imply

$$\begin{aligned}
\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu &\le C\|\iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi) - V_{\varepsilon,\lambda,\xi}\|_\mu \\
&\le C\left(\left\|\iota^*\left(f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi - \left(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right)\right\|_\mu\right. \\
&\qquad \left. + \left\|\iota^*\left(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right) - V_{\varepsilon,\lambda,\xi}\right\|_\mu\right) \\
&\le C\left(\left\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi - \left(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right\|_{2N/(N+2)}\right. \\
&\qquad \left. + \sum_{i=1}^k O(\mu\delta_i) + O\left((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\right)\right) \\
&\le C\big\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_\varepsilon'(V_{\varepsilon,\lambda,\xi})\phi\big\|_{2N/(N+2)} + C\big\|(f_\varepsilon'(V_{\varepsilon,\lambda,\xi}) - f_0'(V_{\varepsilon,\lambda,\xi}))\phi\big\|_{2N/(N+2)} \\
&\qquad + C\big\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_0(V_{\varepsilon,\lambda,\xi})\big\|_{2N/(N+2)} \\
&\qquad + C\left\|f_0(V_{\varepsilon,\lambda,\xi}) - \left(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)\right)\right\|_{2N/(N+2)} \\
&\qquad + \sum_{i=1}^k O(\mu\delta_i) + O\left((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\right).
\end{aligned} \tag{3.23}$$

By using Lemma (A.5) and noticing that

$$\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi) - f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_\varepsilon'(V_{\varepsilon,\lambda,\xi})\phi\|_{2N/(N+2)} \le C\|\phi\|_\mu^{2^*-1},$$

we deduce

$$\begin{aligned}
\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu &\le C\|\phi\|_\mu^{2^*-1} + C\varepsilon\|\phi\|_\mu + C\varepsilon + O(\sigma^{\frac{N+2}{2}}) + \sum_{i=1}^k O\big(\delta_i^{\frac{N+2}{2}}\big) + \sum_{i=1}^k O(\mu\delta_i) + O\big((\mu\sigma^{\frac{N-2}{2}})^{\frac12}\big) \\
&= C\|\phi\|_\mu^{2^*-1} + C\varepsilon\|\phi\|_\mu + O\big(\varepsilon^{\frac{N+2}{2(N-2)}}\big) + O\big(\varepsilon^{\frac{1+2\alpha}{4}}\big).
\end{aligned}$$

The remaining argument can be obtained by standard arguments, see e.g. [4]. □

Now we consider the reduced functional

$$I_\varepsilon(\lambda,\xi) = J_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}).$$

Proposition 3.3. *Let (λ^0,ξ^0) with $\lambda^0 = (\lambda_1^0,\dots,\lambda_k^0,\overline{\lambda}^0)$ and $\xi^0 = (\xi_1^0,\xi_2^0,\dots,\xi_k^0)$ be a critical point of $I_\varepsilon(\lambda,\xi)$. Then there exists a family of solutions to problem (1.1) of the form*

$$u_\varepsilon = V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}.$$

Proof. It is enough to prove that (3.2) holds. Let ∂_s denote one of ∂_{λ_i}, $\partial_{\overline{\lambda}}$, $\partial_{(\xi_i)^j}$, $i=1,\dots,k$, $j=1,\dots,N$. As in [25], equation (3.1) implies:

$$\begin{aligned}\partial_s I_\varepsilon(\lambda,\xi) &= J'_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi})(\partial_s V_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi})\\ &= \sum_{i=1}^k\sum_{j=0}^N c_{i,j}\big(P\Psi_i^j,\partial_s V_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi}\big)+c_0\big(P\overline{\Psi},\partial_s V_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi}\big).\end{aligned}$$

Now it remains to prove that $c_{i,j}=0$ for $i=1,\dots,k$ and $j=0,\dots,N$, and $c_0=0$, provided $\varepsilon>0$ is small enough.

If (λ,ξ) is a critical point of $I_\varepsilon(\lambda,\xi)$, then

$$\sum_{i=1}^k\sum_{j=0}^N c_{i,j}(P\Psi_i^j,\partial_s V_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi})+c_0(P\overline{\Psi},\partial_s V_{\varepsilon,\lambda,\xi}+\partial_s\phi_{\varepsilon,\lambda,\xi})=0. \tag{3.24}$$

Observe that

$$\partial_{\lambda_i}V_{\varepsilon,\lambda,\xi}=-\varepsilon^{\frac{1}{N-2}}P\Psi_i^0,\quad \partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi}=\varepsilon^{\frac{1}{N-2}}P\overline{\Psi},\quad \partial_{(\xi_i)^j}V_{\varepsilon,\lambda,\xi}=-P\Psi_i^j,\quad j=1,\dots,N. \tag{3.25}$$

On the other hand, $(P\Psi_i^j,\phi_{\varepsilon,\lambda,\xi})=0$ for $j=0,1,\dots,N$, Proposition 3.2 and Lemma A.6 imply

$$(P\Psi_i^j,\partial_s\phi_{\varepsilon,\lambda,\xi})=-(\partial_s P\Psi_i^j,\phi_{\varepsilon,\lambda,\xi})=O(\|\partial_s P\Psi_i^j\|_\mu\|\phi_{\varepsilon,\lambda,\xi}\|_\mu)=o(\|\partial_s P\Psi_i^j\|_\mu)=o(\delta_i^{-2}).$$

Similarly,

$$(P\overline{\Psi},\partial_s\phi_{\varepsilon,\lambda,\xi})=o(\|\partial_s P\overline{\Psi}\|_\mu)=o(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}).$$

Now Lemma A.1, (3.24) and (3.25) yield

$$\begin{aligned}0&=\sum_{i=1}^k\sum_{j=0}^N c_{i,j}(P\Psi_i^j,\partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi})+c_0(P\overline{\Psi},\partial_{\overline{\lambda}}V_{\varepsilon,\lambda,\xi})+o\big(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big)\\ &=\varepsilon^{\frac{1}{N-2}}\left(\sum_{i=1}^k\sum_{j=0}^N c_{i,j}(P\Psi_i^j,P\overline{\Psi})+c_0(P\overline{\Psi},P\overline{\Psi})\right)+o\big(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big)\\ &=c_0\widetilde{c}_0\varepsilon^{\frac{1}{N-2}}\sigma^{-2}\big(1+o(1)\big),\end{aligned}$$

which implies $c_0=0$. Similar arguments show that $c_{i,j}=0$ for $i=1,\dots,k$, $j=0,1,\dots,N$. $\square$

3.2 Proofs of Theorems 1.1 and 1.2

In this part, we consider $V_{\varepsilon,\lambda,\xi}=-\sum\limits_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma$. The reduced energy is expanded as follows.

Lemma 3.4. *For $\varepsilon\to 0^+$ there holds*

$$I_\varepsilon(\lambda,\xi)=a_1+a_2\varepsilon-a_3\varepsilon^\alpha-a_4\varepsilon\ln\varepsilon+\psi(\lambda,\xi)\varepsilon+o(\varepsilon) \tag{3.26}$$

C^1-uniformly with respect to (λ,ξ) in compact sets of $\mathcal{O}_\eta$. The constants are given by $a_1=\frac{1}{N}(k+1)S_0^{\frac{N}{2}}$, $a_2=\frac{(k+1)}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}-\frac{k+1}{(2^*)^2}S_0^{\frac{N}{2}}$, $a_3=\frac12 S_0^{\frac{N-2}{2}}\overline{S}\mu_0$, and $a_4=\frac{k+1}{2\cdot 2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}$. The function ψ is given by

$$\begin{aligned}\psi(\lambda,\xi)=b_1(H(0,0)\overline{\lambda}^{N-2}&+\sum_{i=1}^k H(\xi_i,\xi_i)\lambda_i^{N-2}+2\sum_{i=1}^k G(\xi_i,0)\lambda_i^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}\\ &-2\sum_{i,j=1,i<j}^k G(\xi_i,\xi_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}})-b_2\ln(\lambda_1\lambda_2\dots\lambda_k\overline{\lambda})^{\frac{N-2}{2}},\end{aligned}$$

with $b_1=\frac{1}{2}C_0\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}$ *and* $b_2=\frac{1}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}$.

Proof. Observe that

$$\begin{aligned}J_\varepsilon(V_{\varepsilon,\lambda,\xi}) &= \frac{1}{2}\int_\Omega(|\nabla V_{\varepsilon,\lambda,\xi}|^2-\mu\frac{|V_{\varepsilon,\lambda,\xi}|^2}{|x|^2}) &(3.27)\\ &-\frac{1}{2^*}\int_\Omega|V_{\varepsilon,\lambda,\xi}|^{2^*} &(3.28)\\ &+(\frac{1}{2^*}\int_\Omega|V_{\varepsilon,\lambda,\xi}|^{2^*}-\frac{1}{2^*-\varepsilon}\int_\Omega|V_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}). &(3.29)\end{aligned}$$

By Lemma A.7, Lemma A.10, and noticing $\mu=\mu_0\varepsilon^\alpha$, $\varepsilon\to0^+$, we obtain

$$\begin{aligned}(3.27) &= \frac{1}{2}\int_\Omega\left(|\nabla PV_\sigma|^2-\mu\frac{|PV_\sigma|^2}{|x|^2}\right)+\sum_{i=1}^k\left(|\nabla PU_{\delta_i,\xi_i}|^2-\mu\frac{|PU_{\delta_i,\xi_i}|^2}{|x|^2}\right)\\ &\quad-\sum_{i=1}^k\int_\Omega\left(\nabla PV_\sigma\nabla PU_{\delta_i,\xi_i}-\mu\frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2}\right)\\ &\quad+\sum_{i,j=1,i<j}^k\int_\Omega\left(\nabla PU_{\delta_i,\xi_i}\nabla PU_{\delta_j,\xi_j}-\mu\frac{PU_{\delta_i,\xi_i}PU_{\delta_j,\xi_j}}{|x|^2}\right)\\ &= \frac{1}{2}(k+1)S_0^{\frac{N}{2}}-\frac{N}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon^\alpha+\frac{1}{2}C_0\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}\Bigg(-H(0,0)\sigma^{N-2}-\sum_{i=1}^kH(\xi_i,\xi_i)\delta_i^{N-2}\\ &\quad-2\sum_{i=1}^k\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}G(\xi_i,0)+2\sum_{i,j=1,i<j}^kG(\xi_i,\xi_j)\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\Bigg)+o(\varepsilon).\end{aligned}$$

By Lemma A.8 and Lemma A.10, and again observing $\mu=\mu_0\varepsilon^\alpha$, $\varepsilon\to0^+$ we obtain:

$$\begin{aligned}(3.28) &= -\frac{1}{2^*}(k+1)S_0^{\frac{N}{2}}+\frac{N-2}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon^\alpha+C_0\int_{\mathbb{R}^N}U_{1,0}^{2^*-1}\Bigg(H(0,0)\sigma^{N-2}+\sum_{i=1}^kH(\xi_i,\xi_i)\delta_i^{N-2}\\ &\quad+2\sum_{i=1}^k\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}G(\xi_i,0)-2\sum_{i,j=1,i<j}^kG(\xi_i,\xi_j)\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\Bigg)+o(\varepsilon).\end{aligned}$$

Next Lemma A.8, Lemma A.9 and Lemma A.10 yield:

$$\begin{aligned}(3.29) &= -\frac{\varepsilon}{(2^*)^2}\int_\Omega|V_{\varepsilon,\lambda,\xi}|^{2^*}+\frac{\varepsilon}{2^*}\int_\Omega|V_{\varepsilon,\lambda,\xi}|^{2^*}\ln|V_{\varepsilon,\lambda,\xi}|+o(\varepsilon)\\ &= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}}+\frac{\varepsilon}{2^*}(-\frac{N-2}{2}\ln\sigma\cdot\int_{\mathbb{R}^N}V_1^{2^*}-\frac{N-2}{2}\ln(\delta_1\ldots\delta_k)\cdot\int_{\mathbb{R}^N}U_{1,0}^{2^*}\\ &\quad+\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1+k\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0})+o(\varepsilon)\\ &= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}}-\frac{(N-2)\varepsilon}{2\cdot2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\cdot\ln(\delta_1\ldots\delta_k\sigma)\\ &\quad+\frac{(k+1)\varepsilon}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}+o(\varepsilon).\end{aligned}$$

Arguing similarly to Lemma 6.1 in [25], we deduce from Proposition 3.2, (2.6), (2.7), and Lemma A.5,

that

$$
\begin{aligned}
J_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi})-J_\varepsilon(V_{\varepsilon,\lambda,\xi}) &= \frac{1}{2}\|\phi_{\varepsilon,\lambda,\xi}\|_\mu^2+\int_\Omega(\nabla V_{\varepsilon,\lambda,\xi}\nabla\phi_{\varepsilon,\lambda,\xi}-\mu\frac{V_{\varepsilon,\lambda,\xi}\phi_{\varepsilon,\lambda,\xi}}{|x|^2}) \\
&\quad -\frac{1}{2^*-\varepsilon}(\int_\Omega|V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}-|V_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}) \qquad (3.30)\\
&= o(\varepsilon).
\end{aligned}
$$

Now (3.27), (3.28), (3.29) and (3.30) imply (3.26). Actually, (3.26)) also holds C^1-uniformly with respect to (λ,ξ) in compact sets of $\mathcal{O}_\eta$; see for example [25, Lemma 7.1]. We omit the details here. □

Proof of Theorem 1.1. The reduced function $\psi(\lambda,\xi)$ from Lemma 3.4 becomes (here $k=1$):

$$
\psi(\lambda,\xi)=b_1\left(H(0,0)\overline{\lambda}^{N-2}+H(\xi_1,\xi_1)\lambda_1^{N-2}+2G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right)-b_2\ln(\lambda_1\overline{\lambda})^{\frac{N-2}{2}}.
$$

Now the first part is almost the same as the one in [4, Theorem 1]. We therefore omit it here.

Now we prove the second part. The symmetry assumption (S_1) and the principle of symmetric criticality (see e.g. [5, Lemma 2.4]) allow us to consider the constrained function of $\psi(\lambda,\xi)$ as follows:

$$
\overline{\psi}(\lambda,t)=b_1\left(h(0,0)\overline{\lambda}^{N-2}+h(t,t)\lambda_1^{N-2}+2g(t,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right)-b_2\ln(\lambda_1\overline{\lambda})^{\frac{N-2}{2}}.
$$

For $t\in(a,b)\setminus\{0\}$, let

$$
\frac{\partial\overline{\psi}(\lambda,t)}{\partial\lambda_1}=(N-2)b_1\left(h(t,t)\lambda_1^{N-3}+g(t,0)\lambda_1^{\frac{N-4}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right)-\frac{(N-2)b_2}{2\lambda_1}=0, \tag{3.31}
$$

and

$$
\frac{\partial\overline{\psi}(\lambda,t)}{\partial\overline{\lambda}}=(N-2)b_1\left(h(0,0)\overline{\lambda}^{N-3}+g(t,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-4}{2}}\right)-\frac{(N-2)b_2}{2\overline{\lambda}}=0. \tag{3.32}
$$

Then it is easy to obtain a unique $\lambda_1(t)>0$ and a unique $\overline{\lambda}(t)>0$ with

$$
(\lambda_1(t))^{\frac{N-2}{2}}=\sqrt{\frac{1}{h(t,t)+g(t,0)(\frac{h(t,t)}{h(0,0)})^{\frac{1}{2}}}\cdot\frac{b_2}{2b_1}} \tag{3.33}
$$

and

$$
(\overline{\lambda}(t))^{\frac{N-2}{2}}=\sqrt{\frac{1}{h(0,0)+g(t,0)(\frac{h(0,0)}{h(t,t)})^{\frac{1}{2}}}\cdot\frac{b_2}{2b_1}}. \tag{3.34}
$$

Now an easy computation using (3.31) and (3.32) shows that:

$$
\begin{aligned}
\frac{\partial^2\overline{\psi}(\lambda,t)}{\partial\lambda_1^2}\Big|_{\lambda_1=\lambda_1(t),\overline{\lambda}=\overline{\lambda}(t)} &= (N-2)b_1\left((N-3)h(t,t)\lambda_1^{N-4}+\frac{N-4}{2}g(t,0)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right)+\frac{(N-2)b_2}{2\lambda_1^2}\\
&= (N-2)b_1\left((N-2)h(t,t)\lambda_1^{N-4}+\frac{N-2}{2}g(t,0)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2\overline{\psi}(\lambda,t)}{\partial\overline{\lambda}^2}\Big|_{\lambda_1=\lambda_1(t),\overline{\lambda}=\overline{\lambda}(t)} &= (N-2)b_1\left((N-3)h(0,0)\overline{\lambda}^{N-4}+\frac{N-4}{2}g(t,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-6}{2}}\right)+\frac{(N-2)b_2}{2\overline{\lambda}^2}\\
&= (N-2)b_1\left((N-2)h(0,0)\overline{\lambda}^{N-4}+\frac{N-2}{2}g(t,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-6}{2}}\right),\\
\frac{\partial^2\overline{\psi}(\lambda,t)}{\partial\overline{\lambda}\partial\lambda_1}\Big|_{\lambda_1=\lambda_1(t),\overline{\lambda}=\overline{\lambda}(t)} &= \frac{(N-2)^2}{2}b_1g(t,0)\lambda_1^{\frac{N-4}{2}}\overline{\lambda}^{\frac{N-4}{2}}.
\end{aligned}
$$

It follows that the Hessian matrix $D^2_{\lambda_1,\overline{\lambda}}\overline{\psi}(\lambda,t)|_{\lambda_1=\lambda_1(t),\overline{\lambda}=\overline{\lambda}(t)}$ is positively definite and therefore nondegenerate.

Now we consider the reduced function

$$\nu(t) := \overline{\psi}(\lambda_1(t),\overline{\lambda}(t),t) = b_2 - b_2\ln\left(\lambda_1(t)\overline{\lambda}(t)\right)^{\frac{N-2}{2}} = b_2 - b_2\ln\frac{b_2}{2b_1} + b_2\ln\varphi(t),$$

where we used (3.31), (3.32), (3.33), and (3.34). Now observe that

$$\nu'(t) = b_2\frac{\varphi'(t)}{\varphi(t)}, \quad \nu''(t) = b_2\frac{\varphi''(t)\varphi(t)-(\varphi'(t))^2}{\varphi^2(t)},$$

so, using that t^* is a (local) minimum of $\varphi(t)$ in $(a,b)\backslash\{0\}$, then it is also a (local) minimum, and therefore a nondegenerate critical point of $\nu(t)$. □

When $\Omega = B(0,1)$, we consider the existence of solutions with $k+1$ bubbles, one positive and k negative for $k=2,3$. We also show that the idea for $k=2,3$ is not applicable for $k=4$. Notice first that the principle of symmetric criticality (see e.g. [5, Lemma 2.4]) allows us to constrain the problem on $B^{(2)} = \{x = (x_1,x_2,0,\dots,0) : x\in B(0,1)\}$ and then to place the bubbles at

$$\xi_i,\xi_{i+1}\in B^{(2)}, \quad \xi_{i+1} = e^{2\pi\sqrt{-1}/k}\xi_i \text{ for } i=1,\dots,k-1. \tag{3.35}$$

Here we used complex coordinates in $B^{(2)}\subset\mathbb{R}^2\times\{0\}$.

For $x,y,z\in\Omega, x\neq y\neq z$, let

$$\begin{aligned}
\alpha_1(x,y) &= \frac{-G(x,0)+\sqrt{G^2(x,0)+4H(0,0)(H(x,x)-G(x,y))}}{2H(0,0)},\\
\beta_1(x,y) &= H(x,x)-G(x,y)+G(x,0)\alpha_1(x,y),\\
\alpha_2(x,y) &= \frac{-2G(x,0)+\sqrt{4G^2(x,0)+4H(0,0)(H(x,x)-2G(x,y))}}{2H(0,0)},\\
\beta_2(x,y) &= H(x,x)-2G(x,y)+G(x,0)\alpha_2(x,y),\\
\alpha_3(x,y,z) &= \frac{-3G(x,0)+\sqrt{9G^2(x,0)+4H(0,0)(H(x,x)-2G(x,y)-G(x,z))}}{2H(0,0)},\\
\beta_3(x,y,z) &= H(x,x)-2G(x,y)-G(x,z)+G(x,0)\alpha_3(x,y,z).
\end{aligned}$$

Then we have the following lemma.

Lemma 3.5. *Let $\Omega = B(0,1)$ and $k=2,3,4$. If $(\lambda,\xi) = (\lambda_1,\dots,\lambda_k,\overline{\lambda},\xi_1,\dots,\xi_k)$ is a critical point of $\psi(\lambda,\xi)$ such that* (3.35) *holds, then*

$$\lambda_1 = \ldots = \lambda_k. \tag{3.36}$$

Moreover we have for $k=2$:

$$\overline{\lambda}^{\frac{N-2}{2}} = \alpha_1(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}, \quad \lambda_1^{\frac{N-2}{2}} = \sqrt{\frac{1}{\beta_1(\xi_1,\xi_2)}\cdot\frac{b_2}{2b_1}}, \quad H(\xi_1,\xi_1)-G(\xi_1,\xi_2)>0; \tag{3.37}$$

for $k=3$:

$$\overline{\lambda}^{\frac{N-2}{2}} = \alpha_2(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}, \quad \lambda_1^{\frac{N-2}{2}} = \sqrt{\frac{1}{\beta_2(\xi_1,\xi_2)}\cdot\frac{b_2}{2b_1}}, \quad H(\xi_1,\xi_1)-2G(\xi_1,\xi_2)>0; \tag{3.38}$$

and for $k = 4$:

$$\overline{\lambda}^{\frac{N-2}{2}} = \alpha_3(\xi_1,\xi_2,\xi_3)\lambda_1^{\frac{N-2}{2}},\ \lambda_1^{\frac{N-2}{2}} = \sqrt{\frac{1}{\beta_3(\xi_1,\xi_2,\xi_3)}\cdot\frac{b_2}{2b_1}},\ H(\xi_1,\xi_1)-2G(\xi_1,\xi_2)-G(\xi_1,\xi_3)>0. \quad (3.39)$$

Proof. If $k = 2$ and (λ,ξ) is a critical point of $\psi(\lambda,\xi)$, then the equations $\frac{\partial\psi(\lambda,\xi)}{\partial\overline{\lambda}} = \frac{\partial\psi(\lambda,\xi)}{\partial\lambda_1} = \frac{\partial\psi(\lambda,\xi)}{\partial\lambda_2} = 0$, imply

$$H(0,0)\overline{\lambda}^{N-2} + G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} + G(\xi_2,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \quad (3.40)$$

$$H(\xi_1,\xi_1)\lambda_1^{N-2} + G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \quad (3.41)$$

$$H(\xi_2,\xi_2)\lambda_2^{N-2} + G(\xi_2,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} = \frac{b_2}{2b_1}. \quad (3.42)$$

Notice that (3.35) yields

$$H(\xi_1,\xi_1) = H(\xi_2,\xi_2), G(\xi_1,0) = G(\xi_2,0),$$

so (3.41) and (3.42) imply

$$\left(\lambda_2^{\frac{N-2}{2}} - \lambda_1^{\frac{N-2}{2}}\right)\left(H(\xi_1,\xi_1)\left(\lambda_2^{\frac{N-2}{2}} + \lambda_1^{\frac{N-2}{2}}\right) + G(\xi_1,0)\overline{\lambda}^{\frac{N-2}{2}}\right) = 0.$$

Since $H(\xi_1,\xi_1) > 0$, $G(\xi_1,0) > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\overline{\lambda} > 0$, we obtain $\lambda_1 = \lambda_2$, and (3.37) follows by an easy computation.

If $k = 3$ and (λ,ξ) is a critical point of $\psi(\lambda,\xi)$, then

$$H(0,0)\overline{\lambda}^{N-2} + G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} + G(\xi_2,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} + G(\xi_3,0)\lambda_3^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \quad (3.43)$$

$$H(\xi_1,\xi_1)\lambda_1^{N-2} + G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} - G(\xi_1,\xi_3)\lambda_1^{\frac{N-2}{2}}\lambda_3^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \quad (3.44)$$

$$H(\xi_2,\xi_2)\lambda_2^{N-2} + G(\xi_2,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} - G(\xi_2,\xi_3)\lambda_2^{\frac{N-2}{2}}\lambda_3^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \quad (3.45)$$

$$H(\xi_3,\xi_3)\lambda_3^{N-2} + G(\xi_3,0)\lambda_3^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_3)\lambda_1^{\frac{N-2}{2}}\lambda_3^{\frac{N-2}{2}} - G(\xi_2,\xi_3)\lambda_2^{\frac{N-2}{2}}\lambda_3^{\frac{N-2}{2}} = \frac{b_2}{2b_1}. \quad (3.46)$$

The ansatz (3.35) gives

$$H(\xi_1,\xi_1) = H(\xi_2,\xi_2) = H(\xi_3,\xi_3),\ G(\xi_1,0) = G(\xi_2,0) = G(\xi_3,0),\ G(\xi_1,\xi_2) = G(\xi_2,\xi_3) = G(\xi_1,\xi_3).$$

We can see from (3.44) and (3.45) that

$$\left(\lambda_2^{\frac{N-2}{2}} - \lambda_1^{\frac{N-2}{2}}\right)(H(\xi_1,\xi_1)(\lambda_2^{\frac{N-2}{2}} + \lambda_1^{\frac{N-2}{2}}) + G(\xi_1,0)\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_3^{\frac{N-2}{2}}) = 0.$$

Assume that

$$H(\xi_1,\xi_1)\left(\lambda_2^{\frac{N-2}{2}} + \lambda_1^{\frac{N-2}{2}}\right) + G(\xi_1,0)\overline{\lambda}^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_3^{\frac{N-2}{2}} = 0. \quad (3.47)$$

Multiplying this by $\lambda_1^{\frac{N-2}{2}}$ and combing it with (3.44), we obtain

$$-H(\xi_1,\xi_1)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} - G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} - \frac{b_2}{2b_1} = 0$$

which is obviously impossible. Therefore $\lambda_2 = \lambda_1$, and similarly $\lambda_3 = \lambda_1$. Finally, (3.38) can be computed directly.

If $k = 4$ and (λ, ξ) is a critical point of $\psi(\lambda, \xi)$, then

$$H(0,0)\overline{\lambda}^{N-2} + \sum_{i=1}^{4} G(\xi_i, 0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \tag{3.48}$$

$$H(\xi_1,\xi_1)\lambda_1^{N-2} + G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - \sum_{i\neq 1} G(\xi_1,\xi_i)\lambda_1^{\frac{N-2}{2}}\lambda_i^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \tag{3.49}$$

$$H(\xi_2,\xi_2)\lambda_2^{N-2} + G(\xi_2,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - \sum_{i\neq 2} G(\xi_2,\xi_i)\lambda_2^{\frac{N-2}{2}}\lambda_i^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \tag{3.50}$$

$$H(\xi_3,\xi_3)\lambda_3^{N-2} + G(\xi_3,0)\lambda_3^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - \sum_{i\neq 3} G(\xi_3,\xi_i)\lambda_3^{\frac{N-2}{2}}\lambda_i^{\frac{N-2}{2}} = \frac{b_2}{2b_1}, \tag{3.51}$$

$$H(\xi_4,\xi_4)\lambda_4^{N-2} + G(\xi_4,0)\lambda_4^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - \sum_{i\neq 4} G(\xi_4,\xi_i)\lambda_4^{\frac{N-2}{2}}\lambda_i^{\frac{N-2}{2}} = \frac{b_2}{2b_1}. \tag{3.52}$$

The ansatz (3.35) gives

$$H(\xi_1,\xi_1) = H(\xi_2,\xi_2) = H(\xi_3,\xi_3) = H(\xi_4,\xi_4), \quad G(\xi_1,0) = G(\xi_2,0) = G(\xi_3,0) = G(\xi_4,0),$$

and

$$G(\xi_1,\xi_2) = G(\xi_2,\xi_3) = G(\xi_3,\xi_4) = G(\xi_4,\xi_1), \quad G(\xi_1,\xi_3) = G(\xi_2,\xi_4).$$

Then using the same argument as the one in case $k = 3$, (3.49) and (3.51) imply $\lambda_1 = \lambda_3$. Similarly we obtain $\lambda_2 = \lambda_4$. Substituting this into (3.49) and (3.50), we then get $\lambda_1 = \lambda_2$. At last, (3.39) follows by direct computation. □

Proof of Theorem 1.2. For $k = 2$, by (3.35) we can assume $\xi_1 = -\xi_2 = (t, 0, \ldots, 0)$, $0 < t < 1$. Then by (3.36) in Lemma 3.5, the reduced function $\psi(\lambda, t)$ becomes

$$f_1(\lambda_1, \overline{\lambda}, t) = b_1\left(h(0,0)\overline{\lambda}^{N-2} + 2h(t,t)\lambda_1^{N-2} + 4g(t,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - 2g(t,-t)\lambda_1^{N-2}\right) - b_2 \ln\left(\lambda_1^2\overline{\lambda}\right)^{\frac{N-2}{2}}.$$

The remaining argument is the same as the one in [6].

Now we consider the case $k = 3$. As a consequence of (3.35) we may assume $\xi_1 = (t, 0, \ldots, 0)$, $\xi_2 = \left(-\frac{t}{2}, \frac{\sqrt{3}t}{2}, 0, \ldots, 0\right)$, $\xi_3 = \left(-\frac{t}{2}, -\frac{\sqrt{3}t}{2}, 0, \ldots, 0\right)$, $0 < t < 1$. Then Lemma 3.5 allows us to consider the function

$$f_2(\lambda_1, \overline{\lambda}, t) = b_1\left(H(0,0)\overline{\lambda}^{N-2} + 3H(\xi_1,\xi_1)\lambda_1^{N-2} + 6G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} - 6G(\xi_1,\xi_2)\lambda_1^{N-2}\right)$$
$$- b_2 \ln\left(\lambda_1^3\overline{\lambda}\right)^{\frac{N-2}{2}}.$$

Setting

$$\gamma_1(t) := H(\xi_1,\xi_1) - 2G(\xi_1,\xi_2) = \frac{1}{(1-t^2)^{N-2}} - \frac{2}{(\sqrt{3}t)^{N-2}} + \frac{2}{(t^4+t^2+1)^{\frac{N-2}{2}}}$$

and

$$\tau_1(t) := G(\xi_1, 0) = \frac{1}{t^{N-2}} - 1,$$

a direct computation shows that $\gamma_1'(t) > 0$, $\gamma_1(t) \to -\infty$ as $t \to 0^+$, and $\gamma_1(\frac{1}{2}) > 0$. Thus there exists $t^* \in (0, \frac{1}{2})$ such that

$$\gamma_1(t^*) = 0, \quad \gamma_1(t) > 0 \text{ for all } t \in (t^*, 1). \tag{3.53}$$

Then for $t \in (t^*, 1)$ there exist unique $\lambda_1(t)$, $\overline{\lambda}(t)$ such that

$$\frac{\partial f_2(\lambda_1, \overline{\lambda}, t)}{\partial \lambda_1} = 0, \quad \frac{\partial f_2(\lambda_1, \overline{\lambda}, t)}{\partial \overline{\lambda}} = 0, \tag{3.54}$$

where $\lambda_1(t), \overline{\lambda}(t)$ are from (3.38). Moreover, a direct computation shows that

$$\begin{aligned}
\frac{\partial^2 f_2(\lambda_1(t), \overline{\lambda}(t), t)}{\partial \lambda_1^2} &= 3(N-2)b_1\left((N-3)\gamma_1(t)\lambda_1^{N-4} + \frac{N-4}{2}\tau_1(t)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right) + \frac{3(N-2)b_2}{2\lambda_1^2} \\
&= 3(N-2)b_1\left((N-2)\gamma_1(t)\lambda_1^{N-4} + \frac{N-2}{2}\tau_1(t)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}}\right), \\
\frac{\partial^2 f_2(\lambda_1(t), \overline{\lambda}(t), t)}{\partial \overline{\lambda}^2} &= (N-2)b_1\left((N-3)H(0,0)\overline{\lambda}^{N-4} + \frac{3(N-4)}{2}\tau_1(t)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-6}{2}}\right) + \frac{(N-2)b_2}{2\overline{\lambda}^2} \\
&= (N-2)b_1\left((N-2)H(0,0)\overline{\lambda}^{N-4} + \frac{3(N-2)}{2}\tau_1(t)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-6}{2}}\right), \\
\frac{\partial^2 f_2(\lambda_1(t), \overline{\lambda}(t), t)}{\partial \overline{\lambda}\partial \lambda_1} &= \frac{3(N-2)^2}{2}b_1\tau_1(t)\lambda_1^{\frac{N-4}{2}}\overline{\lambda}^{\frac{N-4}{2}}.
\end{aligned}$$

It follows that the Hessian matrix $D^2_{\lambda_1,\overline{\lambda}}f_2(\lambda_1(t), \overline{\lambda}(t), t)$ is positively definite and therefore nondegenerate. Then it is enough to consider the function

$$\nu_1(t) := f_2\left(\lambda_1(t), \overline{\lambda}(t), t\right) = 2b_2 - b_2 \ln\left(\lambda_1^3(t)\overline{\lambda}(t)\right)^{\frac{N-2}{2}}.$$

As in [6, (3.4)] there holds

$$\lim_{t\to(t^*)^+} \nu_1(t) = -\infty \quad \text{and} \quad \lim_{t\to 1^-} \nu_1(t) = +\infty. \tag{3.55}$$

Now we prove $\nu_1'(\frac{1}{2}) < 0$ for N large. Setting

$$\alpha_2(t) := \alpha_2(\xi_1, \xi_2) = -\tau_1(t) + \sqrt{\tau_1^2(t) + \gamma_1(t)},$$

where we used $H(0,0) = 1$, we obtain

$$\nu_1'(t) = \frac{\partial f_2(\lambda_1(t), \overline{\lambda}(t), t)}{\partial t} = 3b_1\left(\gamma_1'(t) + 2\alpha_2(t)\tau_1'(t)\right)\lambda_1^{N-2}.$$

Then by letting $\iota_1(t) := \gamma_1'(t) + 2\alpha_2(t)\tau_1'(t)$, we need to show $\iota_1\left(\frac{1}{2}\right) < 0$ for N large enough. Since $\frac{\gamma_1(\frac{1}{2})}{\tau_1^2(\frac{1}{2})} < 1$ for N large we see as in [6, (3.9)] that

$$\iota_1(\tfrac{1}{2}) \le \gamma_1'(\tfrac{1}{2}) + \frac{4\gamma_1(\frac{1}{2})}{5\tau_1(\frac{1}{2})}\tau_1'(\tfrac{1}{2}).$$

A direct computation gives for N large:

$$\begin{aligned}
\gamma_1'(\tfrac{1}{2}) &= (N-2)\left((\tfrac{4}{3})^{N-1} + 4(\tfrac{2}{\sqrt{3}})^{N-2} - \frac{\frac{3}{2}}{(\frac{1}{16}+\frac{1}{4}+1)^{\frac{N}{2}}}\right) < \frac{11(N-2)}{10}(\tfrac{4}{3})^{N-1}, \\
\tau_1'(\tfrac{1}{2}) &= -(N-2)2^{N-1}, \\
\frac{\gamma_1(\frac{1}{2})}{\tau_1(\frac{1}{2})} &= \frac{(\frac{4}{3})^{N-2} - 2(\frac{2}{\sqrt{3}})^{N-2} + \frac{2}{(\frac{1}{16}+\frac{1}{4}+1)^{\frac{N-2}{2}}}}{2^{N-2}-1} > \frac{11}{12}\cdot\frac{(\frac{4}{3})^{N-2}}{2^{N-2}},
\end{aligned}$$

which yield $\iota_1(\frac{1}{2}) < 0$ for N large enough. Then we have $t_1, t_2 \in (t^*, 1)$, $t_1 \ne t_2$ such that $\nu_1'(t_1) = 0$, $\nu_1'(t_2) = 0$, and we conclude as in [6]. $\square$

Remark 3.6. a) For $k = 3$, $N = 7$, numerical computations show that one cannot find $t_0 \in (t^*, 1)$ such that $\nu_1'(t_0) = 0$. Therefore we can only consider N large enough in this case.

b) For $k = 4$, the idea above also cannot give the existence of solutions with $k+1$ bubbles, one positive at the origin and k negative. In fact, following the above idea, with the ansatz (3.35) we may assume $\xi_1 = (t, 0, \dots, 0)$, $\xi_2 = (0, t, 0, \dots, 0)$, $\xi_3 = (-t, 0, \dots, 0)$, and $\xi_4 = (0, -t, \dots, 0)$, $0 < t < 1$. As a consequence of Lemma 3.5 we need to consider the function

$$f_3(\lambda_1, \overline{\lambda}, t) = b_1 \Big(H(0,0)\overline{\lambda}^{N-2} + 4H(\xi_1, \xi_1)\lambda_1^{N-2} + 8G(\xi_1, 0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} \\ -8G(\xi_1, \xi_2)\lambda_1^{N-2} - 4G(\xi_1, \xi_3)\lambda_1^{N-2}\Big) - b_2 \ln(\lambda_1^4 \overline{\lambda})^{\frac{N-2}{2}}.$$

Now let $\tau_1(t)$ be the same as above and define

$$\begin{aligned} \gamma_2(t) &:= H(\xi_1, \xi_1) - 2G(\xi_1, \xi_2) - G(\xi_1, \xi_3) \\ &= \frac{1}{(1-t^2)^{N-2}} - \frac{2}{(\sqrt{2}t)^{N-2}} + \frac{2}{(t^4+1)^{\frac{N-2}{2}}} - \frac{1}{(2t)^{N-2}} + \frac{1}{(t^2+1)^{N-2}}. \end{aligned}$$

A direct computation shows that

$$\gamma_2'(t) = (N-2)\left(\frac{2t}{(1-t^2)^{N-1}} + \frac{2}{(\sqrt{2})^{N-2}t^{N-1}} - \frac{4t^3}{(t^4+1)^{\frac{N}{2}}} + \frac{1}{2^{N-2}t^{N-1}} - \frac{2t}{(t^2+1)^{N-1}} \right) > 0.$$

Clearly $\gamma_2(t) \to -\infty$ as $t \to 0^+$, and $\gamma_2\left(\frac{1}{\sqrt{2}}\right) > 0$. Then there exists $t^* \in (0, \frac{1}{\sqrt{2}})$ such that

$$\gamma_2(t^*) = 0, \gamma_2(t) > 0, \quad \forall\, t \in (t^*, 1). \tag{3.56}$$

Set $\iota_2(t) := \gamma_2'(t) + 2\alpha_3(t)\tau_1'(t)$, where $\alpha_3(t) := \alpha_3(\xi_1, \xi_2, \xi_3) = \frac{-3\tau_1(t) + \sqrt{9\tau_1^2(t) + 4\gamma_2(t)}}{2}$. If we can prove that $\iota_2(t_0) = 0$ for some $t_0 \in (t^*, 1)$, then problem (1.1) admits a solution with 5 bubbles, one positive at the origin and 4 negative. But the following proposition shows that t_0 does not exist.

Proposition 3.7. *For any $t \in (t^*, 1), N \geq 7$, it always holds that $\iota_2(t) > 0$.*

Proof. We first show that $t^* > \frac{\sqrt{6}-\sqrt{2}}{2}$, where t^* is from (3.56). In order to see that we prove $\gamma_2\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right) < 0$. Since $2^{2/5} \cdot 2(\frac{\sqrt{6}-\sqrt{2}}{2})^2 < 1 < (\frac{\sqrt{6}-\sqrt{2}}{2})^4 + 1$, we have

$$\frac{1}{(\sqrt{2} \cdot \frac{\sqrt{6}-\sqrt{2}}{2})^{N-2}} > \frac{2}{((\frac{\sqrt{6}-\sqrt{2}}{2})^4 + 1)^{\frac{N-2}{2}}}, \quad \text{for all } N \geq 7.$$

On the other hand, it is easy to see that

$$\frac{1}{(1 - (\frac{\sqrt{6}-\sqrt{2}}{2})^2)^{N-2}} = \frac{1}{(\sqrt{2}(\frac{\sqrt{6}-\sqrt{2}}{2}))^{N-2}}, \quad \frac{1}{(2(\frac{\sqrt{6}-\sqrt{2}}{2}))^{N-2}} > \frac{1}{((\frac{\sqrt{6}-\sqrt{2}}{2})^2 + 1)^{N-2}},$$

and we conclude $\gamma_2(\frac{\sqrt{6}-\sqrt{2}}{2}) < 0$.

Now we prove $\iota_2(t) > 0$ for $t \in (t^*, 1) \subset (\frac{\sqrt{6}-\sqrt{2}}{2}, 1)$. It is easy to see that for $t \in \left(\frac{\sqrt{6}-\sqrt{2}}{2}, 1\right)$ there holds

$$\gamma_2'(t) \geq (N-2) \cdot \frac{2t}{(1-t^2)^{N-1}}, \quad \gamma_2(t) \leq \frac{1}{(1-t^2)^{N-2}}.$$

Then for all $t \in (t^*, 1)$ and $N \geq 7$:

$$\begin{aligned}
\frac{\iota_2(t)}{N-2} &\geq \frac{2t}{(1-t^2)^{N-1}} - 3\left(\frac{1}{t^{N-2}} - 1\right) \cdot \frac{1}{t^{N-1}} \left(\sqrt{1 + \frac{4 \cdot \frac{1}{(1-t^2)^{N-2}}}{9(\frac{1}{t^{N-2}} - 1)^2}} - 1 \right) \\
&\geq \frac{2t}{(1-t^2)^{N-1}} - \frac{3}{t^{2N-3}} \cdot \left(\sqrt{1 + \frac{4}{9}(\frac{t^2}{1-t^2})^{N-2} \cdot \frac{1}{(1-t^{N-2})^2}} - 1 \right).
\end{aligned}$$

Setting $T := \frac{t^2}{1-t^2}$ it is enough to prove that

$$\frac{2}{3} \cdot T^{N-1} + 1 > \sqrt{1 + \frac{4}{9} \cdot T^{N-2} \cdot \frac{1}{(1-t^{N-2})^2}},$$

which is equivalent to

$$(T^N + 3T) \cdot (1-t^{N-2})^2 > 1. \tag{3.57}$$

It is easy to see that

$$3T \cdot (1-t^{N-2})^2 \geq 3(1-t^5)^2 > 1 \quad \text{if } t \in [\tfrac{1}{\sqrt{2}}, \tfrac{4}{5}) \tag{3.58}$$

and

$$T^N \cdot (1-t^{N-2})^2 \geq T^N \cdot (1-t)^2 = \frac{t^4}{(1+t)^2} \cdot T^{N-2} > \frac{(\frac{4}{5})^4}{4} (\frac{(\frac{4}{5})^2}{1-(\frac{4}{5})^2})^5 > 1 \text{ if } t \in [\frac{4}{5}, 1). \tag{3.59}$$

Now it is left to prove (3.57) for $t \in \left(t^*, \frac{1}{\sqrt{2}}\right)$. First of all, if $t \in \left(\frac{\sqrt{6}-\sqrt{2}}{2}, \frac{1}{\sqrt{2}}\right)$, then $T \in \left(\frac{\sqrt{3}-1}{2}, 1\right)$. Setting

$$f(T) := 3T(1-t^{N-2})^2 = 3T(1-(\frac{T}{1+T})^{\frac{N-2}{2}})^2,$$

a direct computation shows that

$$\begin{aligned}
f'(T) &= \left(1 - \left(\frac{T}{1+T}\right)^{\frac{N-2}{2}}\right) \left(3 - 3\left(\frac{T}{1+T}\right)^{\frac{N-2}{2}} - 3(N-2)\left(\frac{T}{1+T}\right)^{\frac{N-2}{2}} \frac{1}{1+T}\right) \\
&\geq \left(1 - \left(\frac{1}{2}\right)^{\frac{N-2}{2}}\right) \left(3 - 3\left(\frac{1}{2}\right)^{\frac{N-2}{2}} - 3(N-2)\left(\frac{1}{2}\right)^{\frac{N-2}{2}} \frac{1}{1+\frac{\sqrt{3}-1}{2}}\right) \\
&\geq \left(1 - \left(\frac{1}{2}\right)^{\frac{N-2}{2}}\right) \left(3 - 3\left(\frac{1}{2}\right)^{\frac{5}{2}} - 15\left(\frac{1}{2}\right)^{\frac{5}{2}} \frac{1}{1+\frac{\sqrt{3}-1}{2}}\right) > 0,
\end{aligned}$$

where in the second inequality we use the fact that $3 - 3(\frac{1}{2})^{\frac{N-2}{2}} - 3(N-2)(\frac{1}{2})^{\frac{N-2}{2}} \frac{1}{1+\frac{\sqrt{3}-1}{2}}$ is increasing in N. Now we conclude that

$$f(T) > 3 \cdot \frac{\sqrt{3}-1}{2} \left(1 - \left(\frac{\frac{\sqrt{3}-1}{2}}{1+\frac{\sqrt{3}-1}{2}}\right)^{\frac{5}{2}}\right)^2 > 1 \quad \text{for all } N \geq 7. \tag{3.60}$$

Combining (3.57), (3.58), (3.59), and (3.60), the proof is finished. □

Remark 3.8. Let $\mu = \mu_0 \varepsilon^\alpha$, $\mu_0 > 0$, $\alpha > \frac{N-4}{N-2}$, N large enough. For $k = 2, 3$, we can also consider smooth bounded domains with the following symmetry condition.

(S_2) If $(\widetilde{x}, x') \in \Omega \subset \mathbb{R}^2 \times \mathbb{R}^{N-2}$ then $(e^{2\pi\sqrt{-1}/k}\widetilde{x}, x') \in \Omega$, and $(\widetilde{x}, -x') \in \Omega$.

We assume further that

(S_3) The map $\varphi_k : \Omega^{(2)} := \{\widetilde{x} \in \mathbb{R}^2 : (\widetilde{x}, 0) \in \Omega\} \to \mathbb{R}$, defined by

$$\varphi_k(\widetilde{x}) := \frac{\alpha_{k-1}\left((\widetilde{x},0),(e^{2\pi\sqrt{-1}/k}\widetilde{x},0)\right)}{\beta_{k-1}^{\frac{k+1}{2}}\left((\widetilde{x},0),(e^{2\pi\sqrt{-1}/k}\widetilde{x},0)\right)},$$

admits a nondegenerate critical point at $\widetilde{\xi}^* \in \Omega^{(2)}$ such that

$$H\left((\widetilde{\xi}^*,0),(\widetilde{\xi}^*,0)\right) - (k-1)G\left((\widetilde{\xi}^*,0),(-\widetilde{\xi}^*,0)\right) > 0.$$

Then for $k = 2, 3$, and using the same methods as in Theorem 1.2, one can show that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a pair of solutions $\pm u_\varepsilon$ to problem (1.1) satisfying

$$u_\varepsilon(x) = C_\mu \left(\frac{\sigma^\varepsilon}{(\sigma^\varepsilon)^2|x|^{\beta_1} + |x|^{\beta_2}}\right)^{\frac{N-2}{2}} - C_0 \sum_i^k \left(\frac{\delta^\varepsilon}{(\delta^\varepsilon)^2 + |x - \xi_i^\varepsilon|^2}\right)^{\frac{N-2}{2}} + o(1),$$

where $\delta^\varepsilon = \lambda^\varepsilon \varepsilon^{\frac{1}{N-2}}$, $\xi_i^\varepsilon = (e^{2\pi i\sqrt{-1}/k}\widetilde{\xi}^\varepsilon, 0)$, $\widetilde{\xi}^\varepsilon \in \Omega^{(2)}$, $\sigma^\varepsilon = \overline{\lambda}^\varepsilon \varepsilon^{\frac{1}{N-2}}$, and for some $\eta > 0$ small enough, $\eta < |\widetilde{\xi}^\varepsilon| < 1 - \eta$, $\lambda^\varepsilon, \overline{\lambda}^\varepsilon \in (\eta, \frac{1}{\eta})$. Moreover $\widetilde{\xi}^\varepsilon \to \widetilde{\xi}^*$ as $\varepsilon \to 0$.

The question is what kind of domain, besides $B(0,1)$, satisfies the assumption (S_3)?

3.3 Proof of Theorem 1.3

In this part, we turn to solutions of the form $V_{\varepsilon,\lambda,\xi} = \sum_{i=1}^k (-1)^i PU_{\delta_i,\xi_i} + PV_\sigma$. Then the reduced function in Lemma 3.4 becomes

$$\begin{aligned}\widetilde{\psi}(\lambda,\xi) = b_1 \Bigg(& H(0,0)\overline{\lambda}^{N-2} + \sum_{i=1}^k H(\xi_i,\xi_i)\lambda_i^{N-2} + 2\sum_{i=1}^k (-1)^{i-1} G(\xi_i,0)\lambda_i^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}} \\ & + 2\sum_{i,j=1,i<j}^k (-1)^{i+j-1} G(\xi_i,\xi_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}}\Bigg) - b_2 \ln(\lambda_1\lambda_2\ldots\lambda_k\overline{\lambda})^{\frac{N-2}{2}},\end{aligned}$$

where b_1, b_2 are as in Lemma 3.4.

Proof of Theorem 1.3. Here we have $k = 4$. Due to the ansatz (3.35) we may assume $\xi_1 = (t, 0, \ldots, 0)$, $\xi_2 = (0, t, 0, \ldots, 0)$, $\xi_3 = (-t, 0, 0, \ldots, 0)$, $\xi_4 = (0, -t, 0, \ldots, 0)$, $0 < t < 1$. It is obvious that

$$H(\xi_1,\xi_1) = H(\xi_2,\xi_2) = H(\xi_3,\xi_3) = H(\xi_4,\xi_4), \quad G(\xi_1,0) = G(\xi_2,0) = G(\xi_3,0) = G(\xi_4,0),$$

and

$$G(\xi_1,\xi_2) = G(\xi_2,\xi_3) = G(\xi_3,\xi_4) = G(\xi_4,\xi_1), \quad G(\xi_1,\xi_3) = G(\xi_2,\xi_4).$$

As in the proof of Lemma 3.5 we have

$$\lambda_1 = \lambda_3, \lambda_2 = \lambda_4,$$

which allows us to consider the function

$$\begin{aligned} f_4(\lambda_1,\lambda_2,\overline{\lambda},t) \;=\; & b_1 \Bigg(H(0,0)\overline{\lambda}^{N-2} + 2H(\xi_1,\xi_1)(\lambda_1^{N-2} + \lambda_2^{N-2}) + 4G(\xi_1,0)\left(\lambda_1^{\frac{N-2}{2}} - \lambda_2^{\frac{N-2}{2}}\right)\overline{\lambda}^{\frac{N-2}{2}} \\ & + 8G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} - 2G(\xi_1,\xi_3)\lambda_1^{N-2} - 2G(\xi_1,\xi_3)\lambda_2^{N-2}\Bigg) - b_2 \ln\left(\lambda_1^2\lambda_2^2\overline{\lambda}\right)^{\frac{N-2}{2}}.\end{aligned}$$

Now suppose $\nabla_{\lambda_1,\lambda_2,\overline{\lambda}} f_4(\lambda_1,\lambda_2,\overline{\lambda},t)=0$. Then we have

$$\begin{aligned} H(0,0)\overline{\lambda}^{N-2}+2G(\xi_1,0)(\lambda_1^{\frac{N-2}{2}}-\lambda_2^{\frac{N-2}{2}})\overline{\lambda}^{\frac{N-2}{2}} &= \frac{b_2}{2b_1}, && (3.61)\\ (H(\xi_1,\xi_1)-G(\xi_1,\xi_3))\lambda_1^{N-2}+G(\xi_1,0)\lambda_1^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}+2G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} &= \frac{b_2}{2b_1}, && (3.62)\\ (H(\xi_1,\xi_1)-G(\xi_1,\xi_3))\lambda_2^{N-2}-G(\xi_1,0)\lambda_2^{\frac{N-2}{2}}\overline{\lambda}^{\frac{N-2}{2}}+2G(\xi_1,\xi_2)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}} &= \frac{b_2}{2b_1}. && (3.63) \end{aligned}$$

From (3.62) and (3.63) we deduce

$$\lambda_2^{\frac{N-2}{2}}-\lambda_1^{\frac{N-2}{2}}=\frac{G(\xi_1,0)}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)}\overline{\lambda}^{\frac{N-2}{2}}, \tag{3.64}$$

which combined with (3.61) implies:

$$\overline{\lambda}^{N-2}=\frac{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)-2G^2(\xi_1,0)}\cdot\frac{b_2}{2b_1}. \tag{3.65}$$

As a consequence of (3.64) we get

$$\lambda_1^{N-2}+\lambda_2^{N-2}-2\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}=\left(\frac{G(\xi_1,0)}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)}\right)^2\overline{\lambda}^{N-2},$$

and then (3.62) and (3.63) yield:

$$\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}=\frac{1}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)+2G(\xi_1,\xi_2)}\cdot\frac{b_2}{2b_1} \tag{3.66}$$

and

$$\lambda_1^{N-2}+\lambda_2^{N-2}=\frac{1}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)+2G(\xi_1,\xi_2)}\cdot\frac{b_2}{b_1}+\left(\frac{G(\xi_1,0)}{H(\xi_1,\xi_1)-G(\xi_1,\xi_3)}\right)^2\overline{\lambda}^{N-2}. \tag{3.67}$$

Let $\tau_1(t)$ be as above, and set

$$\gamma_3(t):=H(\xi_1,\xi_1)-G(\xi_1,\xi_3)=\frac{1}{(1-t^2)^{N-2}}-\frac{1}{(2t)^{N-2}}+\frac{1}{(t^2+1)^{N-2}}$$

and

$$\gamma_4(t):=G(\xi_1,\xi_2)=\frac{1}{(\sqrt{2}t)^{N-2}}-\frac{1}{(t^4+1)^{\frac{N-2}{2}}}.$$

A direct computation shows that $\gamma_3'(t)>0$, $\gamma_3(t)\to-\infty$ as $t\to 0^+$, $\gamma_3(t)\to+\infty$ as $t\to 1^-$, and $\gamma_3(\frac{1}{2})>0$. Thus there exists $t_1^*\in(0,\frac{1}{2})$ such that

$$\gamma_3(t_1^*)=0,\quad \gamma_3(t)<0 \text{ for all } t\in(0,t_1^*).$$

On the other hand, $(\gamma_3(t)-2\tau_1^2(t))'>0$, $\gamma_3(t)-2\tau_1^2(t)\to-\infty$ as $t\to 0^+$, $\gamma_3(t)-2\tau_1^2(t)\to+\infty$ as $t\to 1^-$, and $\gamma_3(\frac{1}{2})-2\tau_1^2(\frac{1}{2})<0$. Thus there exists $t_2^*\in(\frac{1}{2},1)$ such that

$$\gamma_3(t_2^*)-2\tau_1^2(t_2^*)=0,\quad \gamma_3(t)-2\tau_1^2(t)>0 \text{ for all } t\in(t_2^*,1).$$

It follows that for every $t\in(0,t_1^*)\cup(t_2^*,1)$ there exist unique $\lambda_1(t)$, $\lambda_2(t)$, $\overline{\lambda}(t)$ such that

$$\nabla_{\lambda_1,\lambda_2,\overline{\lambda}} f_4(\lambda_1(t),\lambda_2(t),\overline{\lambda}(t),t)=0,$$

where $\lambda_1(t), \lambda_2(t), \overline{\lambda}(t)$ satisfy (3.64), (3.65), (3.66) and (3.67). Moreover, a direct computation using (3.61), (3.62), and (3.63) shows that

$$\begin{aligned}
\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\lambda_1^2} &= (N-2)b_1\Big(2(N-3)\gamma_3(t)\lambda_1^{N-4} + (N-4)\tau_1(t)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}} \\
&\quad +2(N-4)\gamma_4(t)\lambda_1^{\frac{N-6}{2}}\lambda_2^{\frac{N-2}{2}}\Big) + \frac{(N-2)b_2}{\lambda_1^2} \\
&= (N-2)^2 b_1\Big(2\gamma_3(t)\lambda_1^{N-4} + \tau_1(t)\lambda_1^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}} + 2\gamma_4(t)\lambda_1^{\frac{N-6}{2}}\lambda_2^{\frac{N-2}{2}}\Big),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\lambda_2^2} &= (N-2)b_1\Big(2(N-3)\gamma_3(t)\lambda_2^{N-4} - (N-4)\tau_1(t)\lambda_2^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}} \\
&\quad +2(N-4)\gamma_4(t)\lambda_1^{\frac{N-6}{2}}\lambda_2^{\frac{N-2}{2}}\Big) + \frac{(N-2)b_2}{\lambda_2^2} \\
&= (N-2)^2 b_1\Big(2\gamma_3(t)\lambda_2^{N-4} - \tau_1(t)\lambda_2^{\frac{N-6}{2}}\overline{\lambda}^{\frac{N-2}{2}} + 2\gamma_4(t)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-6}{2}}\Big),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\overline{\lambda}^2} &= (N-2)b_1\Big((N-3)H(0,0)\overline{\lambda}^{N-4} + (N-4)\tau_1(t)(\lambda_1^{\frac{N-2}{2}} - \lambda_2^{\frac{N-2}{2}})\overline{\lambda}^{\frac{N-6}{2}}\Big) \\
&\quad + \frac{(N-2)b_2}{2\overline{\lambda}^2} \\
&= (N-2)^2 b_1\Big(H(0,0)\overline{\lambda}^{N-4} + \tau_1(t)(\lambda_1^{\frac{N-2}{2}} - \lambda_2^{\frac{N-2}{2}})\overline{\lambda}^{\frac{N-6}{2}}\Big),
\end{aligned}$$

$$\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\overline{\lambda}\partial\lambda_1} = (N-2)^2 b_1\tau_1(t)\lambda_1^{\frac{N-4}{2}}\overline{\lambda}^{\frac{N-4}{2}},$$

$$\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\overline{\lambda}\partial\lambda_2} = -(N-2)^2 b_1\tau_1(t)\lambda_2^{\frac{N-4}{2}}\overline{\lambda}^{\frac{N-4}{2}},$$

$$\frac{\partial^2 f_4(\lambda_1,\lambda_2,\overline{\lambda},t)}{\partial\lambda_1\partial\lambda_2} = 2(N-2)^2 b_1\gamma_4(t)\lambda_1^{\frac{N-4}{2}}\lambda_2^{\frac{N-4}{2}}.$$

For simplicity, we introduce the notation

$$X := \overline{\lambda}^{\frac{N-2}{2}}, \quad Y := \lambda_1^{\frac{N-2}{2}}, \quad Z := \lambda_2^{\frac{N-2}{2}}.$$

In order to show that the Hessian matrix $D^2_{\lambda_1,\lambda_2,\overline{\lambda}} f_4(\lambda_1,\lambda_2,\overline{\lambda},t)$ is nondegenerate for any $t \in (0,t_1^*)\cup(t_2^*,1)$, it suffices to show that the matrix

$$\begin{pmatrix}
X + \tau_1(t)(Y-Z) & \tau_1(t)Y^{\frac{2}{N-2}}X^{\frac{N-4}{N-2}} & -\tau_1(t)Z^{\frac{2}{N-2}}X^{\frac{N-4}{N-2}} \\
\tau_1(t)Y^{\frac{N-4}{N-2}}X^{\frac{2}{N-2}} & 2\gamma_3(t)Y + \tau_1(t)X + 2\gamma_4(t)Z & 2\gamma_4(t)Y^{\frac{N-4}{N-2}}Z^{\frac{2}{N-2}} \\
-\tau_1(t)Z^{\frac{N-4}{N-2}}X^{\frac{2}{N-2}} & 2\gamma_4(t)Y^{\frac{2}{N-2}}Z^{\frac{N-4}{N-2}} & 2\gamma_3(t)Z - \tau_1(t)X + 2\gamma_4(t)Y
\end{pmatrix}$$

is nondegenerate. Using (3.61), (3.62) and (3.63) this is equivalent to show that the matrix

$$\begin{pmatrix}
\frac{X}{2} + \frac{b_2}{4b_1}\cdot\frac{1}{X} & \tau_1(t)Y^{\frac{2}{N-2}}X^{\frac{N-4}{N-2}} & -\tau_1(t)Z^{\frac{2}{N-2}}X^{\frac{N-4}{N-2}} \\
\tau_1(t)Y^{\frac{N-4}{N-2}}X^{\frac{2}{N-2}} & \gamma_3(t)Y + \frac{b_2}{2b_1}\cdot\frac{1}{Y} & 2\gamma_4(t)Y^{\frac{N-4}{N-2}}Z^{\frac{2}{N-2}} \\
-\tau_1(t)Z^{\frac{N-4}{N-2}}X^{\frac{2}{N-2}} & 2\gamma_4(t)Y^{\frac{2}{N-2}}Z^{\frac{N-4}{N-2}} & \gamma_3(t)Z + \frac{b_2}{2b_1}\cdot\frac{1}{Z}
\end{pmatrix}$$

is nondegenerate. A direct computation, using (3.66), shows that the determinant of the above matrix has the same sign as $\gamma_3(t)$, and hence is nondegenerate.

Now in order to finish the proof, we look for $t_0 \in (0, t_1^*)$ such that $\nu_2'(t_0) = 0$, where

$$\nu_2(t) := f_4\big(\lambda_1(t), \lambda_2(t), \overline{\lambda}(t), t\big).$$

Observe that

$$\begin{aligned}\nu_2'(t) &= \frac{\partial f_4(\lambda_1(t), \lambda_2(t), \overline{\lambda}(t), t)}{\partial t}\\ &= 2b_1\left(\gamma_3'(t)\left(\lambda_1^{N-2} + \lambda_2^{N-2}\right) + 2\tau_1'(t)\left(\lambda_1^{\frac{N-2}{2}} - \lambda_2^{\frac{N-2}{2}}\right)\overline{\lambda}^{\frac{N-2}{2}} + 4\gamma_4'(t)\lambda_1^{\frac{N-2}{2}}\lambda_2^{\frac{N-2}{2}}\right)\end{aligned}$$

where $\lambda_1, \lambda_2, \overline{\lambda}$ satisfy (3.64), (3.65), (3.66) and (3.67). Therefore, $\nu_2'(t) = 0$ for $t \in (0, t_1^*)$ is equivalent to

$$\begin{aligned}\iota_3(t) :=\ &\gamma_3'(t)\big(2\gamma_3(t)(\gamma_3(t) - 2\tau_1^2(t)) + \tau_1^2(t)(\gamma_3(t) + 2\gamma_4(t))\big) - 2\tau_1'(t)\tau_1(t)\gamma_3(t)\big(\gamma_3(t) + 2\gamma_4(t)\big)\\ &+ 4\gamma_4'(t)\gamma_3(t)\big(\gamma_3(t) - 2\tau_1^2(t)\big)\\ =\ &0.\end{aligned}$$

It is easy to check that $\iota_3(t) \to -\infty$ as $t \to 0^+$ and $\iota_3(t_1^*) > 0$ since $\gamma_3'(t_1^*) > 0$, $\gamma_4(t_1^*) > 0$ and $\gamma_3(t_1^*) = 0$. Hence there exists $t_0 \in (0, t_1^*)$ such that $\iota_3(t_0) = 0$, which finishes the proof. $\square$

Remark 3.9. It seems that there also should exist $t_0 \in (t_2^*, 1)$ such that $\iota_3(t_0) = 0$. This is not considered here because the computations get enormous.

4 Solutions with tower of bubbles concentrating at the origin

In this section we prove Theorem 1.4 where $\alpha = 1$. We use the same notations in similar settings as Section 3.

4.1 The finite dimensional reduction

We fix an integer $k \geq 0$. For $\lambda = (\lambda_1, \dots, \lambda_k, \overline{\lambda}) \in \mathbb{R}^{k+1}$ we set $\delta_i = \lambda_i \varepsilon^{\frac{2i-1}{N-2}}$, for $i = 1, \dots, k$, and consider $\zeta = (\zeta_1, \dots, \zeta_k) \in (\mathbb{R}^N)^k$ such that

$$\xi = (\xi_1, \dots, \xi_k) = (\delta_1\zeta_1, \dots, \delta_k\zeta_k) \in \Omega^k.$$

We also set $\sigma = \overline{\lambda}\varepsilon^{\frac{2(k+1)-1}{N-2}}$. Now we define for $\eta \in (0,1)$:

$$\mathcal{O}_\eta = \left\{(\lambda, \zeta) \in \mathbb{R}_+^{k+1} \times (\mathbb{R}^N)^k : \lambda_i \in (\eta, \eta^{-1}),\ \overline{\lambda} \in (\eta, \eta^{-1}),\ |\zeta_i| \leq \frac{1}{\eta},\ i = 1, \dots, k\right\}.$$

We also recall the sets $W_{\varepsilon,\lambda,\xi}$, $K_{\varepsilon,\lambda,\xi}$, $K_{\varepsilon,\lambda,\xi}^\perp$, and the projections $\Pi_{\varepsilon,\lambda,\xi}$, $\Pi_{\varepsilon,\lambda,\xi}^\perp$ as in Section 3. Now we want to find $\eta > 0$,$\varepsilon > 0$, $(\lambda, \zeta) \in \mathcal{O}_\eta$ and $\phi_{\varepsilon,\lambda,\xi} \in K_{\varepsilon,\lambda,\xi}^\perp$ such that:

$$\Pi_{\varepsilon,\lambda,\xi}^\perp\left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}))\right) = 0, \tag{4.1}$$

$$\Pi_{\varepsilon,\lambda,\xi}\left(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi} - \iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}))\right) = 0, \tag{4.2}$$

where

$$V_{\varepsilon,\lambda,\xi} = \sum_{i=1}^k (-1)^{i-1} PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma. \tag{4.3}$$

Next we take $\rho > 0$ small enough and let

$$A_{k+1} := B(0, \sqrt{\delta_{k+1}\delta_k}), \quad A_i := B(0, \sqrt{\delta_i \delta_{i-1}}) \setminus B(0, \sqrt{\delta_i \delta_{i+1}}) \text{ for } i = 1, \dots, k; \tag{4.4}$$

here $\delta_0 = \frac{\rho^2}{\delta_1}$, $\delta_{k+1} = \sigma$; cf. [25]. Finally recall the operator $L_{\varepsilon,\lambda,\xi}$ from Section 3.

Now we first solve (4.1).

Proposition 4.1. *For any $\eta > 0$, there exist $\varepsilon_0 > 0$ and $c > 0$ such that for every $(\lambda, \zeta) \in \mathcal{O}_\eta$, and for every $\varepsilon \in (0, \varepsilon_0)$:*

$$\|L_{\varepsilon,\lambda,\xi}(\phi)\|_\mu \geq c\|\phi\|_\mu \quad \text{for all } \phi \in K^\perp_{\varepsilon,\lambda,\xi}. \tag{4.5}$$

Consequently, $L_{\varepsilon,\lambda,\xi}$ is invertible with continuous inverse.

Proof. Arguing by contradiction, we assume that there exist $\eta > 0$, sequences $\varepsilon^n > 0$, $(\lambda^n, \zeta^n) \in \mathcal{O}_\eta$, $\phi^n \in H_\mu(\Omega)$ with $\varepsilon^n \to 0$, $\lambda^n_i \to \lambda_i$, $\overline{\lambda}^n \to \overline{\lambda}$, $\zeta^n_i \to \zeta_i$, as $n \to \infty$ and such that

$$\phi^n \in K^\perp_{\varepsilon^n,\lambda^n,\xi^n}, \quad \|\phi^n\|_\mu = 1, \tag{4.6}$$

and

$$L_{\varepsilon^n,\lambda^n,\xi^n}(\phi^n) = h^n \ \text{ with } \|h^n\|_\mu \to 0; \tag{4.7}$$

here $\lambda^n = (\lambda^n_1, \dots, \lambda^n_k, \overline{\lambda}^n)$, $\zeta^n = (\zeta^n_1, \dots, \zeta^n_k)$, $\xi^n = (\xi^n_1, \dots, \xi^n_k) = (\delta^n_1\zeta^n_1, \delta^n_2\zeta^n_2, \dots, \delta^n_k\zeta^n_k) \in \Omega^k$, $\delta^n_i = \lambda^n_i \varepsilon^{\frac{2i-1}{N-2}}$ for $i = 1, 2, \dots, k$, $\sigma^n = \overline{\lambda}^n \varepsilon^{\frac{2(k+1)-1}{N-2}}$. We need the sets

$$A^n_{k+1} := B\left(0, \sqrt{\delta^n_{k+1}\delta^n_k}\right), \quad A^n_i := B\left(0, \sqrt{\delta^n_i\delta^n_{i-1}}\right) \setminus B\left(0, \sqrt{\delta^n_i\delta^n_{i+1}}\right), \ i = 1, 2, \dots, k,$$

where $\delta^n_0 := \frac{\rho^2}{\delta^n_1}$, $\delta^n_{k+1} := \sigma^n$.

Thus we have:

$$\phi^n - \iota^*\left(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n\right) = h^n - \Pi_{\varepsilon^n,\lambda^n,\xi^n}\left(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)\right). \tag{4.8}$$

Then we obtain as in Proposition 3.1

$$w^n := -\Pi_{\varepsilon^n,\lambda^n,\xi^n}(\iota^*(f_0'(V_{\varepsilon^n,\lambda^n,\xi^n})\phi^n)) = \sum_{i=1}^{k}\sum_{j=0}^{N} c^n_{i,j} P(\Psi^j_i)_n + c^n_0 P(\overline{\Psi})_n$$

for some coefficients $c^n_{i,j}$, c^n_0, where $(\Psi^j_i)_n$, $j = 1, \dots, N, (\Psi^0_i)_n$, and $(\overline{\Psi})_n$ are defined as in the proof of Proposition 3.1.

Step 1. We claim that

$$\lim_{n\to\infty} \|w^n\|_\mu = 0. \tag{4.9}$$

Multiplying (4.8) by $\Delta P(\Psi^h_l)_n + \mu\frac{P(\Psi^h_l)_n}{|x|^2}$, using Lemma B.1, Lemma A.2, Lemma B.2, and arguing as in the proof of Proposition 3.1, we deduce $c^n_{l,h} \to 0$, for $l = 1, \dots, k$, $h = 0, 1, \dots, N$, and $c^n_0 \to 0$, as $n \to \infty$. The claim $\lim\limits_{n\to\infty} \|w^n\|_\mu = 0$ follows.

Step 2. As in [25], we use cut-off functions χ^n_i, $i = 1, \dots, k+1$, with the properties

$$\begin{cases} \chi^n_i(x) = 1 \quad \text{if } \sqrt{\delta^n_i\delta^n_{i+1}} \leq |x| \leq \sqrt{\delta^n_i\delta^n_{i-1}}; \\ \chi^n_i(x) = 0 \quad \text{if } |x| \leq \dfrac{\sqrt{\delta^n_i\delta^n_{i+1}}}{2} \text{ or } |x| \geq 2\sqrt{\delta^n_i\delta^n_{i-1}}; \\ |\nabla\chi^n_i(x)| \leq \dfrac{1}{\sqrt{\delta^n_i\delta^n_{i-1}}} \quad \text{and} |\nabla^2\chi^n_i(x)| \leq \dfrac{4}{\delta^n_i\delta^n_{i-1}}, \end{cases}$$

for $i = 1, \dots, k$, and

$$\begin{cases} \chi_{k+1}^n(x) = 1, \quad \text{if } |x| \le \sqrt{\delta_{k+1}^n \delta_k^n}; \\ \chi_{k+1}^n(x) = 0, \quad \text{if } |x| \ge 2\sqrt{\delta_{k+1}^n \delta_k^n}; \\ |\nabla \chi_{k+1}^n(x)| \le \dfrac{1}{\sqrt{\delta_{k+1}^n \delta_k^n}}, \quad \text{and} \quad |\nabla^2 \chi_{k+1}^n(x)| \le \dfrac{4}{\delta_{k+1}^n \delta_k^n}. \end{cases}$$

The function ϕ_i^n defined by

$$\phi_i^n(y) := (\delta_i^n)^{\frac{N-2}{2}} \phi^n(\delta_i^n y) \chi_i^n(\delta_i^n y), \quad \text{for } y \in \Omega_i^n := \frac{\Omega}{\delta_i^n}, \ i = 1, \dots, k+1.$$

is bounded in $D^{1,2}(\mathbb{R}^N)$. Therefore we may assume, up to a subsequence,

$$\phi_i^n \rightharpoonup \phi_i^\infty \ \text{ weakly in } D^{1,2}(\mathbb{R}^N), \ i = 1, 2, \dots, k+1.$$

Now we prove

$$\phi_i^\infty = 0 \quad \text{for } i = 1, \dots, k+1. \tag{4.10}$$

As in Proposition 3.1, using (4.8), (4.7), (4.9), we have for any $\psi \in C_0^\infty(\mathbb{R}^N)$, and for $i = 1, \dots, k$:

$$\begin{aligned} \int_{\Omega_i^n} \nabla \phi_i^n(y) \nabla \psi(y) &= (\delta_i^n)^{\frac{2-N}{2}} \int_\Omega \nabla \iota^* \left(f_0'(V_{\varepsilon^n, \lambda^n, \xi^n}(x)) \phi^n(x) \right) \nabla \left(\chi_i^n(x) \psi\left(\frac{x}{\delta_i^n} \right) \right) + o(1) \\ &= (\delta_i^n)^{\frac{2-N}{2}} \int_\Omega f_0'\left(V_{\varepsilon^n, \lambda^n, \xi^n}(x)\right) \phi^n(x) \chi_i^n(x) \psi\left(\frac{x}{\delta_i^n} \right) + o(1) \\ &= (\delta_i^n)^2 \int_{\Omega_i^n} f_0'\left(V_{\varepsilon^n, \lambda^n, \xi^n}(\delta_i^n y)\right) \phi_i^n(y) \psi(y) + o(1) \\ &= \int_{\mathbb{R}^N} f_0'\left(U_{1,\zeta_i}(y)\right) \phi_i^\infty(y) \psi(y) + o(1). \end{aligned}$$

Hence ϕ_i^∞ is a weak solution of

$$-\Delta \phi_i^\infty = f_0'(U_{1,\zeta_i}) \phi_i^\infty, \ \text{in } D^{1,2}(\mathbb{R}^N).$$

Setting $\Psi_{1,\zeta_i}^j := \frac{\partial U_{1,\zeta_i}}{\partial (\zeta_i)^j}$, for $j = 1, \dots, N$, and $\Psi_{1,\zeta_i}^0 := \frac{\partial U_{\delta,\zeta_i}}{\partial \delta}|_{\delta=1}$ we obtain as in [25, Lemma 3.1]:

$$\int_{\mathbb{R}^N} \nabla \phi_i^\infty(x) \nabla \Psi_{1,\zeta_i}^j(x) = 0, \ j = 0, 1, 2, \dots, N, \ i = 1, 2, \dots, k.$$

Then (4.10) holds for $i = 1, \dots, k$. The proof of $\phi_{k+1}^\infty = 0$ is similar.

Step 3. A contradiction arises as in Proposition 3.1 and [24]. □

Proposition 4.2. *For any $\eta > 0$, there exist $\varepsilon_0 > 0$, $c_0 > 0$ such that for every $(\lambda, \zeta) \in \mathcal{O}_\eta$ and every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique solution $\phi_{\varepsilon,\lambda,\xi} \in K_{\varepsilon,\lambda,\xi}^\perp$ of equation (4.1). Moreover, we have*

$$\|\phi_{\varepsilon,\lambda,\xi}\|_\mu \le c_0 (\varepsilon^{\frac{N+2}{2(N-2)}} + \varepsilon^{\frac{2k+3}{4}}), \tag{4.11}$$

and the map $\Phi_\varepsilon : \mathcal{O}_\eta \to K_{\varepsilon,\lambda,\xi}^\perp$ defined by $\Phi_\varepsilon(\lambda, \xi) := \phi_{\varepsilon,\lambda,\xi}$ is of class C^1.

Proof. As in [4], we define the operator $T_{\varepsilon,\lambda,\xi} : K_{\varepsilon,\lambda,\xi}^\perp \to K_{\varepsilon,\lambda,\xi}^\perp$ by

$$T_{\varepsilon,\lambda,\xi}(\phi) = L_{\varepsilon,\lambda,\xi}^{-1} \Pi_{\varepsilon,\lambda,\xi}^\perp (\iota^*(f_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi) - f_0'(V_{\varepsilon,\lambda,\xi})\phi) - V_{\varepsilon,\lambda,\xi}).$$

Now we prove that $T_{\varepsilon,\lambda,\xi}$ is a contraction mapping. Proposition 4.1, (2.2) and Lemma B.3 imply as in (3.23):

$$\begin{aligned}\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu \le{}& C\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi)-f_\varepsilon(V_{\varepsilon,\lambda,\xi})-f'_\varepsilon(V_{\varepsilon,\lambda,\xi})\phi\|_{2N/(N+2)}\\ &+C\|(f'_\varepsilon(V_{\varepsilon,\lambda,\xi})-f'_0(V_{\varepsilon,\lambda,\xi}))\phi\|_{2N/(N+2)}\\ &+C\|f_\varepsilon(V_{\varepsilon,\lambda,\xi})-f_0(V_{\varepsilon,\lambda,\xi})\|_{2N/(N+2)}\\ &+C\left\|f_0(V_{\varepsilon,\lambda,\xi})-\left(\sum_{i=1}^k(-1)^{i-1}f_0(U_{\delta_i,\xi_i})+(-1)^kf_0(V_\sigma)\right)\right\|_{2N/(N+2)}\\ &+\sum_{i=1}^kO(\mu\delta_i)+O\left(\left(\mu\sigma^{\frac{N-2}{2}}\right)^{\frac12}\right).\end{aligned}$$

Using Lemma (B.4) and observing that

$$\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi)-f_\varepsilon(V_{\varepsilon,\lambda,\xi})-f'_\varepsilon(V_{\varepsilon,\lambda,\xi})\phi\|_{2N/(N+2)}\le C\|\phi\|_\mu^{2^*-1},$$

we have

$$\begin{aligned}\|T_{\varepsilon,\lambda,\xi}(\phi)\|_\mu &\le C\|\phi\|_\mu^{2^*-1}+C\varepsilon\|\phi\|_\mu+C\varepsilon+O\left(\varepsilon^{\frac{N+2}{2(N-2)}}\right)+\sum_{i=1}^kO(\mu\delta_i)+O\left(\left(\mu\sigma^{\frac{N-2}{2}}\right)^{\frac12}\right)\\ &=C\|\phi\|_\mu^{2^*-1}+C\varepsilon\|\phi\|_\mu+O\left(\varepsilon^{\frac{N+2}{2(N-2)}}\right)+O\left(\varepsilon^{\frac{2k+3}{4}}\right).\end{aligned}$$

The remaining part of the argument is standard. □

For $\lambda=(\lambda_1,\dots,\lambda_k,\overline{\lambda})$ and $\zeta=(\zeta_1,\dots,\zeta_k)$ we now consider the reduced functional

$$I_\varepsilon(\lambda,\zeta)=J_\varepsilon(V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi}).$$

Proposition 4.3. *If (λ^0,ζ^0) is a critical point of I_ε then there exists a family of solutions u_ε to problem* (1.1) *having the shape*

$$u_\varepsilon(x)=V_{\varepsilon,\lambda,\xi}+\phi_{\varepsilon,\lambda,\xi},\tag{4.12}$$

where $V_{\varepsilon,\lambda,\xi}$ is the one stated in (4.3).

Proof. We omit the proof because it is similarly to the one of Proposition 3.3. □

4.2 Proof of Theorem 1.4

For convenience, we use the notation $\lambda_{k+1}:=\overline{\lambda}$ in this subsection.

Lemma 4.4. *For $\varepsilon\to0^+$, there holds*

$$I_\varepsilon(\lambda,\zeta)\;=\;a_1+a_2\varepsilon-a_3\varepsilon\ln\varepsilon+\psi(\lambda,\zeta)\varepsilon+o(\varepsilon)\tag{4.13}$$

C^1-uniformly with respect to (λ,ζ) in compact sets of $\mathcal{O}_\eta$. The constants are given by $a_1=\frac{k+1}{N}S_0^{\frac N2}$, $a_2=\frac{(k+1)}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}-\frac{k+1}{(2^*)^2}S_0^{\frac N2}-\frac12S_0^{\frac{N-2}{2}}\overline{S}\mu_0$, and $a_3=\frac{(k+1)^2}{2\cdot2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}$. The function ψ is given by

$$\psi(\lambda,\zeta)=b_1\lambda_1^{N-2}+\sum_{i=1}^kb_2(\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}}h_1(\zeta_i)-\sum_{i=1}^kb_3h_2(\zeta_i)-b_4\ln(\lambda_1\dots\lambda_{k+1})^{\frac{N-2}{2}},$$

with $b_1 = \frac{1}{2}C_0 \int_{\mathbb{R}^N} U_{1,0}^{2^*-1}$, $b_2 = C_0^{2^*}$, $b_3 = \frac{1}{2}C_0^2\mu_0$, $b_4 = \frac{1}{2^*}\int_{\mathbb{R}^N} U_{1,0}^{2^*}$, *and*

$$h_1(\zeta_i) = \int_{\mathbb{R}^N} \frac{1}{|y+\zeta_i|^{N-2}(1+|y|^2)^{\frac{N+2}{2}}}, \quad h_2(\zeta_i) = \int_{\mathbb{R}^N} \frac{1}{|y+\zeta_i|^2(1+|y|^2)^{N-2}}.$$

Proof. Observe that

$$\begin{aligned} J_\varepsilon(V_{\varepsilon,\lambda,\xi}) &= \frac{1}{2}\int_\Omega \left(|\nabla V_{\varepsilon,\lambda,\xi}|^2 - \mu\frac{|V_{\varepsilon,\lambda,\xi}|^2}{|x|^2}\right) && (4.14)\\ &\quad -\frac{1}{2^*}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*} && (4.15)\\ &\quad +\left(\frac{1}{2^*}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*} - \frac{1}{2^*-\varepsilon}\int_\Omega |V_{\varepsilon,\lambda,\xi}|^{2^*-\varepsilon}\right). && (4.16)\end{aligned}$$

For $k \geq 1$ Lemma B.5 and Lemma A.10 yield

$$\begin{aligned}(4.14) = {} & \frac{1}{2}(k+1)S_0^{\frac{N}{2}} - \frac{N}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon - \frac{1}{2}C_0^{2^*}H(0,0)\lambda_1^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\cdot\varepsilon\\ & - \frac{1}{2}\sum_{i=1}^k \mu_0 C_0^2\int_{\mathbb{R}^N}\frac{1}{|y|^2(1+|y-\zeta_i|^2)^{N-2}}\cdot\varepsilon\\ & - C_0^{2^*}\left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\cdot\frac{1}{(1+|\zeta_k|^2)^{\frac{N-2}{2}}}\cdot\varepsilon\\ & - \sum_{i=1}^{k-1} C_0^{2^*}\left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\cdot\frac{1}{(1+|\zeta_i|^2)^{\frac{N-2}{2}}}\cdot\varepsilon + o(\varepsilon).\end{aligned}$$

From Lemma B.6 and Lemma A.10 we deduce:

$$\begin{aligned}(4.15) = {} & -\frac{1}{2^*}(k+1)S_0^{\frac{N}{2}} + \frac{N-2}{4}S_0^{\frac{N-2}{2}}\overline{S}\mu_0\varepsilon + C_0^{2^*}H(0,0)\lambda_1^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\cdot\varepsilon\\ & + C_0^{2^*}\sum_{i=1}^k\left(\frac{\lambda_{i+1}}{\lambda_i}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{|y|^{N-2}(1+|y-\zeta_i|^2)^{\frac{N+2}{2}}}\cdot\varepsilon\\ & + C_0^{2^*}\sum_{i=1}^k(\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|y|^2)^{\frac{N+2}{2}}}\frac{1}{(1+|\zeta_i|^2)^{\frac{N-2}{2}}}\cdot\varepsilon + o(\varepsilon).\end{aligned}$$

By Lemma B.7 and Lemma A.10,

$$\begin{aligned}(4.16) &= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}} - \frac{(N-2)\varepsilon}{2\cdot 2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\cdot\ln(\delta_1\ldots\delta_k\sigma)\\ &\quad + \frac{(k+1)\varepsilon}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0} + o(\varepsilon)\\ &= -\frac{\varepsilon}{(2^*)^2}(k+1)S_0^{\frac{N}{2}} - \frac{(N-2)\varepsilon}{2\cdot 2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\cdot\ln(\lambda_1\ldots\lambda_k\overline{\lambda})\\ &\quad - \frac{(k+1)^2}{2\cdot 2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\cdot\varepsilon\ln\varepsilon + \frac{(k+1)}{2^*}\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}\cdot\varepsilon + o(\varepsilon).\end{aligned}$$

Using Proposition 4.2, (2.6), (2.7)), Lemma B.4, we get:

$$J_\varepsilon(V_{\varepsilon,\lambda,\xi} + \phi_{\varepsilon,\lambda,\xi}) - J_\varepsilon(V_{\varepsilon,\lambda,\xi}) = o(\varepsilon). \tag{4.17}$$

Now we conclude the proof for $k \geq 1$ by (4.14), (4.15),(4.16), and (4.17).

The case $k = 0$ can be easily dealt with using (A.27), (A.47) and (4.17). Observe here that (4.13) holds C^1-uniformly with respect to (λ, ζ) in compact sets of $\mathcal{O}_\eta$; see [25, Lemma 7.1]. $\square$

Proof of Theorem 1.4. By the change of variables

$$\lambda_1^{\frac{N-2}{2}} = s_1, \quad \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{N-2}{2}} = s_2, \ \dots, \ \left(\frac{\lambda_{k+1}}{\lambda_k}\right)^{\frac{N-2}{2}} = s_{k+1},$$

$\psi(\lambda, \zeta)$ can be rewritten as

$$\widehat{\psi}(s, \zeta) = b_1 s_1^2 + \sum_{i=1}^k b_2 s_{i+1} h_1(\zeta_i) - \sum_{i=1}^k b_3 h_2(\zeta_i) - b_4 \ln(s_1^{k+1} s_2^k \dots s_{k+1}),$$

where $s = (s_1, s_2, \dots, s_{k+1})$.

Suppose $\frac{\widehat{\psi}(s,\zeta)}{\partial s_i} = 0$ for $i = 1, \dots, k+1$, and let $\widehat{s}(\zeta) = (\widehat{s}_1(\zeta), \dots, \widehat{s}_{k+1}(\zeta))$ be the corresponding critical point. Then

$$\widehat{s}_1 = \sqrt{\frac{(k+1)b_4}{2b_1}}, \quad \widehat{s}_2 = \frac{k b_4}{b_2 h_1(\zeta_1)}, \ \dots, \ \widehat{s}_{k+1} = \frac{b_4}{b_2 h_1(\zeta_k)},$$

and it is easy to show that $\widehat{s}(\zeta)$ is non-degenerate. Plugging these into $\widehat{\psi}(s, \zeta)$ gives

$$\begin{aligned} \widehat{\psi}(\widehat{s}(\zeta), \zeta) = & \frac{(k+1)^2 b_4}{2} - \sum_{i=1}^k b_3 h_2(\zeta_i) - b_4 (\frac{k+1}{2} \ln \frac{(k+1)b_4}{2b_1} \\ & + \sum_{i=1}^k i \ln \frac{i b_4}{b_2}) + \sum_{i=1}^k b_4 (k+1-i) \ln h_1(\zeta_i) \\ = & C_1 + \sum_{i=1}^k g_i(\zeta_i), \end{aligned} \tag{4.18}$$

where

$$C_1 = \frac{(k+1)^2 b_4}{2} - b_4 \left(\frac{k+1}{2} \ln \frac{(k+1)b_4}{2b_1} + \sum_{i=1}^k i \ln \frac{i b_4}{b_2} \right),$$

and

$$g_i(\zeta_i) = b_4 (k+1-i) \ln \int_{\mathbb{R}^N} \frac{1}{|y+\zeta_i|^{N-2} (1+|y|^2)^{\frac{N+2}{2}}} - b_3 \int_{\mathbb{R}^N} \frac{1}{|y+\zeta_i|^2 (1+|y|^2)^{N-2}}.$$

A direct computation shows that $\zeta_i = 0$ is a critical point of $g_i(\zeta_i)$ such that

$$\frac{\partial^2 g_i(\zeta_i)}{\partial \zeta_i^j \partial \zeta_i^l}|_{\zeta_i = 0} = 0 \text{ if } j \neq l;$$

and

$$\frac{\partial^2 g_i(\zeta_i)}{\partial (\zeta_i^j)^2}|_{\zeta_i = 0} = \frac{2N-8}{N} \int_{\mathbb{R}^N} \frac{b_3}{|y|^4 (1+|y|^2)^{N-2}} > 0.$$

Consequently $\zeta_i = 0$ is a nondegenerate local minimum of g_i, hence $\zeta = 0$ is a C^1-stable critical point of $\widehat{\psi}(\widehat{s}(\zeta), \zeta)$. In particular, small C^1-perturbations of $\widehat{\psi}(\widehat{s}(\zeta), \zeta)$ still have a critical point, close to 0. Thus we conclude the proof. $\square$

Acknowledgements. The authors would like to thank Prof. Daomin Cao for many helpful discussions during the preparation of this paper. This work was carried out while Qianqiao Guo was visiting Justus-Liebig-Universität Gießen, to which he would like to express his gratitude for their warm hospitality.

A Appendix

In this part, we give the lemmas used in Section 3.

Take $0<\eta<\min\{|\xi_i|, \mathrm{dist}(\xi_i,\partial\Omega), |\xi_{i_1}-\xi_{i_2}|, i, i_1, i_2=1,2,\dots,k\}$. Similarly to Lemma A.5 in [25], we obtain the following.

Lemma A.1. *For $i,l=1,2,\dots,k$, and $j,h=0,1,\dots,N$, it holds*

$$\begin{aligned}
(P\overline{\Psi},P\overline{\Psi}) &= \widetilde{c}_0\frac{1}{\sigma^2}+o(\frac{1}{\sigma^2}); && \text{(A.1)}\\
(P\overline{\Psi},P\Psi_i^j) &= o(\frac{1}{\sigma^2})(\text{and } o(\frac{1}{\delta_i^2})); && \text{(A.2)}\\
(P\Psi_i^j,P\Psi_i^j) &= \widetilde{c}_{i,j}\frac{1}{\delta_i^2}+o(\frac{1}{\delta_i^2}); && \text{(A.3)}\\
(P\Psi_i^j,P\Psi_l^h) &= o(\frac{1}{\delta_i^2})(\text{and } o(\frac{1}{\delta_l^2})) \quad \text{if } i\neq l \text{ or } j\neq h, && \text{(A.4)}
\end{aligned}$$

where $\widetilde{c}_0>0, \widetilde{c}_{i,j}>0$ are constants.

Proof. We only prove (A.1) and (A.2) as $j=0$. (A.2) as $j\neq 0$ is similar. (A.3) and (A.4) are from Lemma A.5 in [25].

To prove (A.1), noticing that $\overline{\Psi}$ is an eigenfunction to (2.3) with $\Lambda=2^*-1$, by Proposition (2.2), we have

$$\begin{aligned}
(P\overline{\Psi},P\overline{\Psi}) &= \int_\Omega |\nabla P\overline{\Psi}|^2-\mu\frac{|P\overline{\Psi}|^2}{|x|^2}\\
&= \int_\Omega \nabla\overline{\Psi}\nabla P\overline{\Psi}-\mu\frac{\overline{\Psi}P\overline{\Psi}}{|x|^2}-\mu\frac{(P\overline{\Psi}-\overline{\Psi})P\overline{\Psi}}{|x|^2}\\
&= (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}^2-(2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}(\overline{\Psi}-P\overline{\Psi})-\mu\frac{(P\overline{\Psi}-\overline{\Psi})P\overline{\Psi}}{|x|^2}\\
&= (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}^2+O(\sigma^{\frac{N-2}{2}})\\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_\Omega \frac{\sigma^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^2}\cdot\sigma^{N-4}\frac{(|x|^{\beta_2}-\sigma^2|x|^{\beta_1})^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^N}+O(\sigma^{\frac{N-2}{2}})\\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_{\frac{\Omega}{\sigma^{\frac{\sqrt{\mu}}{\sqrt{\overline{\mu}}-\mu}}}} \frac{1}{\sigma^2}\frac{(|y|^{\beta_2}-|y|^{\beta_1})^2}{(|y|^{\beta_1}+|y|^{\beta_2})^{2+N}}+O(\sigma^{\frac{N-2}{2}}) \quad (x=\sigma^{\frac{\sqrt{\mu}}{\sqrt{\overline{\mu}}-\mu}}y)\\
&= \frac{(N^2-4)C_\mu^{2^*}}{4}\int_{\mathbb{R}^N} \frac{1}{\sigma^2}\frac{(|y|^{\beta_2}-|y|^{\beta_1})^2}{(|y|^{\beta_1}+|y|^{\beta_2})^{2+N}}+o(\frac{1}{\sigma^2})\\
&= \widetilde{c}_0\frac{1}{\sigma^2}+o(\frac{1}{\sigma^2}),
\end{aligned}$$

for a positive constant $\widetilde{c}_0$. Similarly,

$$
\begin{aligned}
&(P\overline{\Psi}, P\Psi_i^0) = \int_\Omega \nabla P\overline{\Psi} \cdot \nabla P\Psi_i^0 - \mu \frac{P\overline{\Psi} \cdot P\Psi_i^0}{|x|^2} \\
= \quad & \int_\Omega \nabla\overline{\Psi}\nabla P\Psi_i^0 - \mu\frac{\overline{\Psi}P\Psi_i^0}{|x|^2} - \mu\frac{(P\overline{\Psi}-\overline{\Psi})P\Psi_i^0}{|x|^2} \\
= \quad & (2^*-1)\int_\Omega V_\sigma^{2^*-2}\overline{\Psi}\Psi_i^0 + O(\sigma^{\frac{N-2}{2}}) \\
= \quad & C_0 C_\mu^{2^*-1}(2^*-1)\int_\Omega \frac{\sigma^2}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^2}\cdot\frac{N-2}{2}\sigma^{\frac{N-4}{2}}\frac{(|x|^{\beta_2}-\sigma^2|x|^{\beta_1})}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N}{2}}} \\
& \cdot\frac{N-2}{2}\delta_i^{\frac{N-4}{2}}\frac{(|x-\xi_i|^2-\delta_i^2)}{(\delta_i^2+|x-\xi_i|^2)^{\frac{N}{2}}} + O(\sigma^{\frac{N-2}{2}}) \\
= \quad & o(\frac{1}{\sigma^2})(\text{and } o(\frac{1}{\delta_i^2})).
\end{aligned}
$$

□

Lemma A.2. *For $i = 1, 2, \ldots, k$, and $j = 0, 1, \ldots, N$, it holds*

$$
\|P\Psi_i^j - \Psi_i^j\|_{2N/(N-2)} = \begin{cases} O(\delta_i^{\frac{N-2}{2}}) & \text{if } j = 1, 2, \ldots, N, \\ O(\delta_i^{\frac{N-4}{2}}) & \text{if } j = 0; \end{cases} \tag{A.5}
$$

$$
\|P\overline{\Psi} - \overline{\Psi}\|_{2N/(N-2)} = O(\sigma^{\frac{N-4}{2}}). \tag{A.6}
$$

Proof. (A.5) can be proved as Lemma B.4 in [24]. (A.6) can be obtained similarly by using Proposition 2.2. □

Lemma A.3.

$$
\|(f_0'(-\sum_{i=1}^k PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h\|_{2N/(N+2)} \tag{A.7}
$$

$$
\leq \begin{cases} O(\sigma^{\frac{N-2}{2}}) + \sum\limits_{i=1}^k O(\delta_i^{\frac{N-2}{2}}) & \text{if } h = 1, 2, \ldots, N, \\ O(\sigma^{\frac{N-2}{2}}) + \sum\limits_{i=1, i\neq l}^k O(\delta_i^{\frac{N-2}{2}}) + O(\delta_l^{\frac{N-4}{2}}) & \text{if } h = 0; \end{cases}
$$

$$
\|(f_0'(-\sum_{i=1}^k PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(V_\sigma))\overline{\Psi}\|_{2N/(N+2)} \tag{A.8}
$$

$$
\leq O(\sigma^{\frac{N-4}{2}}) + \sum_{i=1}^k O(\delta_i^{\frac{N-2}{2}}).
$$

Proof. We only prove (A.7).

$$\begin{aligned}
&\int_{\Omega} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
= & \int_{B(\xi_l,\frac{\eta}{2})} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
&+ \int_{B(0,\frac{\eta}{2})\bigcup\bigcup\limits_{i=1,i\neq l}^{k} B(\xi_i,\frac{\eta}{2})} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
&+ \int_{\Omega\setminus(B(0,\frac{\eta}{2})\bigcup\bigcup\limits_{i=1}^{k} B(\xi_i,\frac{\eta}{2}))} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)}.
\end{aligned}$$

First of all, by (2.4), (2.7),

$$\begin{aligned}
&\int_{B(\xi_l,\frac{\eta}{2})} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
\leq & \int_{B(\xi_l,\frac{\eta}{2})} |(f_0'(PU_{\delta_l,\xi_l}) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} + O(\sigma^{\frac{N(N-2)}{N+2}}) + \sum_{i=1,i\neq l}^{k} O(\delta_i^{\frac{N(N-2)}{N+2}}) \\
\leq & \ O(\sigma^{\frac{N(N-2)}{N+2}}) + \sum_{i=1}^{k} O(\delta_i^{\frac{N(N-2)}{N+2}}).
\end{aligned}$$

For $i \neq l$, we have

$$\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{2})} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
= & \int_{B(\xi_i,\frac{\eta}{2})} |(f_0'(PU_{\delta_i,\xi_i}) + O(\sigma^{\frac{N-2}{2}}) + \sum_{j=1,j\neq i,j\neq l}^{k} O(\delta_j^{\frac{N-2}{2}}) + O(\delta_l^2))\Psi_l^h|^{2N/(N+2)} \\
= & \begin{cases} O(\delta_l^{\frac{N(N-2)}{N+2}}) & \text{if } \ h = 1,2,\dots,N, \\ O(\delta_l^{\frac{N(N-4)}{N+2}}) & \text{if } \ h = 0. \end{cases}
\end{aligned}$$

At last,

$$\begin{aligned}
&\int_{\Omega\setminus B(0,\frac{\eta}{2})\bigcup\limits_{i=1}^{k} B(\xi_i,\frac{\eta}{2})} |(f_0'(-\sum_{i=1}^{k} PU_{\delta_i,\xi_i} + PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
\leq & \begin{cases} O(\delta_l^{\frac{N(N-2)}{N+2}})(O(\sigma^{\frac{4N}{N+2}}) + \sum\limits_{i=1}^{k} O(\delta_i^{\frac{4N}{N+2}})) & \text{if } \ h = 1,2,\dots,N, \\ O(\delta_l^{\frac{N(N-4)}{N+2}})(O(\sigma^{\frac{4N}{N+2}}) + \sum\limits_{i=1}^{k} O(\delta_i^{\frac{4N}{N+2}})) & \text{if } \ h = 0. \end{cases}
\end{aligned}$$

Then (A.7) follows. □

Lemma A.4.

$$\|\iota^*(-\sum_{i=1}^{k} f_0(U_{\delta_i,\xi_i}) + f_0(V_\sigma)) - V_{\varepsilon,\lambda,\xi}\|_\mu \leq \sum_{i=1}^{k} O(\mu\delta_i) + O((\mu\sigma^{\frac{N-2}{2}})^{\frac{1}{2}}). \tag{A.9}$$

Proof. By Definition (2.1), there holds

$$\begin{aligned}&\int_\Omega \nabla \iota^*(f_0(V_\sigma))\nabla(\iota^*(f_0(V_\sigma))-PV_\sigma)-\mu\int_\Omega \frac{\iota^*(f_0(V_\sigma))(\iota^*(f_0(V_\sigma))-PV_\sigma)}{|x|^2} \\ = &\int_\Omega f_0(V_\sigma)(\iota^*(f_0(V_\sigma))-PV_\sigma).\end{aligned} \tag{A.10}$$

It also has

$$\begin{cases}-\Delta PV_\sigma=-\Delta V_\sigma=\mu\frac{V_\sigma}{|x|^2}+f_0(V_\sigma) & \text{in } \Omega,\\ PV_\sigma=0 & \text{on } \partial\Omega.\end{cases}$$

Then

$$\begin{aligned}&\int_\Omega \nabla PV_\sigma\nabla(\iota^*(f_0(V_\sigma))-PV_\sigma)-\mu\int_\Omega \frac{V_\sigma(\iota^*(f_0(V_\sigma))-PV_\sigma)}{|x|^2} \\ = &\int_\Omega f_0(V_\sigma)(\iota^*(f_0(V_\sigma))-PV_\sigma).\end{aligned} \tag{A.11}$$

Combining with (A.10) and (A.11) it holds

$$\int_\Omega |\nabla(\iota^*(f_0(V_\sigma))-PV_\sigma)|^2=\mu\int_\Omega \frac{(\iota^*(f_0(V_\sigma))-V_\sigma)(\iota^*(f_0(V_\sigma))-PV_\sigma)}{|x|^2}. \tag{A.12}$$

By (2.4),

$$\|\iota^*(f_0(V_\sigma))-PV_\sigma\|_\mu=(\mu\int_\Omega \frac{|(V_\sigma-PV_\sigma)(\iota^*(f_0(V_\sigma))-PV_\sigma)|}{|x|^2})^{\frac12}\le O((\mu\sigma^{\frac{N-2}{2}})^{\frac12}). \tag{A.13}$$

Similarly to (A.10), (A.11), we also have

$$\begin{aligned}&\int_\Omega \nabla\iota^*(f_0(U_{\delta_i,\xi_i}))\nabla(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}) \\ &-\mu\int_\Omega \frac{\iota^*(f_0(U_{\delta_i,\xi_i}))(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2} \\ = &\int_\Omega f_0(U_{\delta_i,\xi_i})(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})\end{aligned} \tag{A.14}$$

and

$$\int_\Omega \nabla PU_{\delta_i,\xi_i}\nabla(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})=\int_\Omega f_0(U_{\delta_i,\xi_i})(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}). \tag{A.15}$$

Then

$$\int_\Omega |\nabla(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})|^2=\mu\int_\Omega \frac{\iota^*(f_0(U_{\delta_i,\xi_i}))(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2}. \tag{A.16}$$

Therefore, by Hölder's inequality and Hardy's inequality,

$$\begin{aligned}\|\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu &= (\mu\int_\Omega \frac{PU_{\delta_i,\xi_i}(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})}{|x|^2})^{\frac12} \\ &\le (\mu(\int_\Omega \frac{(PU_{\delta_i,\xi_i})^2}{|x|^2})^{\frac12}(\int_\Omega \frac{(\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i})^2}{|x|^2})^{\frac12})^{\frac12} \\ &\le C(\mu\delta_i\|\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu)^{\frac12},\end{aligned}$$

which implies

$$\|\iota^*(f_0(U_{\delta_i,\xi_i}))-PU_{\delta_i,\xi_i}\|_\mu\le O(\mu\delta_i). \tag{A.17}$$

Hence, (A.9) follows from (A.13) and (A.17). $\square$

Lemma A.5.

$$\|(f'_\varepsilon(V_{\varepsilon,\lambda,\xi})-f'_0(V_{\varepsilon,\lambda,\xi}))\phi\|_{2N/(N+2)} = C\varepsilon\|\phi\|_\mu; \tag{A.18}$$

$$\|f_\varepsilon(V_{\varepsilon,\lambda,\xi})-f_0(V_{\varepsilon,\lambda,\xi})\|_{2N/(N+2)} = C\varepsilon; \tag{A.19}$$

$$\|f_0(V_{\varepsilon,\lambda,\xi})-(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma))\|_{2N/(N+2)} = O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}). \tag{A.20}$$

Proof. (A.18) and (A.19) can be seen in [4].

By (2.6), (2.7),

$$\begin{aligned}
&(\int_{B(0,\frac{\eta}{2})}|f_0(V_{\varepsilon,\lambda,\xi})-(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma))|^{2N/(N+2)})^{(N+2)/2N}\\
\leq\ & C(\int_{B(0,\frac{\eta}{2})}|(PV_\sigma)^{2^*-1}-V_\sigma^{2^*-1}|^{2N/(N+2)})^{(N+2)/2N}+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}})\\
\leq\ & C\sigma^{\frac{N-2}{2}}(\int_{B(0,\frac{\eta}{2})}|(V_\sigma)^{2^*-2}|^{2N/(N+2)})^{(N+2)/2N}+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}})\\
=\ & O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}),
\end{aligned}$$

$$\begin{aligned}
&(\int_{B(\xi_i,\frac{\eta}{2})}|f_0(V_{\varepsilon,\lambda,\xi})-(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma))|^{2N/(N+2)})^{(N+2)/2N}\\
\leq\ & C(\int_{B(\xi_i,\frac{\eta}{2})}|(PU_{\delta_i,\xi_i})^{2^*-1}-U_{\delta_i,\xi_i}^{2^*-1}|^{2N/(N+2)})^{(N+2)/2N}+O(\sigma^{\frac{N+2}{2}})+\sum_{j=1,j\neq i}^k O(\delta_j^{\frac{N+2}{2}})\\
=\ & O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}),
\end{aligned}$$

$$\begin{aligned}
&(\int_{\Omega\setminus(B(0,\frac{\eta}{2})\bigcup_{i=1}^k B(\xi_i,\frac{\eta}{2}))}|f_0(V_{\varepsilon,\lambda,\xi})-(-\sum_{i=1}^k f_0(U_{\delta_i,\xi_i})+f_0(V_\sigma))|^{2N/(N+2)})^{(N+2)/2N}\\
=\ & O(\sigma^{\frac{N+2}{2}})+\sum_{i=1}^k O(\delta_i^{\frac{N+2}{2}}),
\end{aligned}$$

then we deduce (A.20). □

Lemma A.6. *For $i=1,2,\dots,k$, there hold*

$$\|\partial_{\lambda_i}P\Psi_i^j\|_\mu = O(\varepsilon^{\frac{1}{N-2}}\delta_i^{-2}),\ j=1,2,\dots,N; \tag{A.21}$$

$$\|\partial_{(\xi_i)^j}P\Psi_i^j\|_\mu = O(\delta_i^{-2}),\ j=1,2,\dots,N; \tag{A.22}$$

$$\|\partial_{(\xi_i)^l}P\Psi_i^j\|_\mu = O(\delta_i^{-2}),\ j,l=1,2,\dots,N, j\neq l; \tag{A.23}$$

$$\|\partial_{\lambda_i}P\Psi_i^0\|_\mu = O(\varepsilon^{\frac{1}{N-2}}\delta_i^{-2}); \tag{A.24}$$

$$\|\partial_{(\xi_i)^j}P\Psi_i^0\|_\mu = O(\delta_i^{-2}),\ j=1,2,\dots,N; \tag{A.25}$$

$$\|\partial_{\overline{\lambda}}P\overline{\Psi}\|_\mu = O(\varepsilon^{\frac{1}{N-2}}\sigma^{-2}). \tag{A.26}$$

Proof. We only prove (A.21) and (A.26) here.

$$\begin{aligned}\|\partial_{\lambda_i} P\Psi_i^j\|_\mu^2 &= \varepsilon^{\frac{2}{N-2}}\|\partial_{\delta_i} P\Psi_i^j\|_\mu^2\\ &\le C\varepsilon^{\frac{2}{N-2}}\int_\Omega \nabla\partial_{\delta_i}\Psi_i^j\nabla\partial_{\delta_i}P\Psi_i^j\\ &= C\varepsilon^{\frac{2}{N-2}}\int_\Omega((2^*-1)(2^*-2)U_{\delta_i,\xi_i}^{2^*-3}\Psi_i^j\Psi_i^0+(2^*-1)U_{\delta_i,\xi_i}^{2^*-2}\partial_{\delta_i}\Psi_i^j)\partial_{\delta_i}P\Psi_i^j\\ &= O(\varepsilon^{\frac{2}{N-2}}\delta_i^{-4}),\end{aligned}$$

then (A.21) is obtained.

$$\begin{aligned}\|\partial_{\overline{\lambda}}P\overline{\Psi}\|_\mu^2 &= \varepsilon^{\frac{2}{N-2}}\|\partial_\sigma P\overline{\Psi}\|_\mu^2\\ &= \varepsilon^{\frac{2}{N-2}}\int_\Omega \nabla\partial_\sigma\overline{\Psi}\nabla\partial_\sigma P\overline{\Psi}-\mu\frac{\partial_\sigma\overline{\Psi}\partial_\sigma P\overline{\Psi}}{|x|^2}+\mu\frac{\partial_\sigma P\overline{\Psi}(\partial_\sigma\overline{\Psi}-\partial_\sigma P\overline{\Psi})}{|x|^2}\\ &= \varepsilon^{\frac{2}{N-2}}(\int_\Omega((2^*-1)(2^*-2)V_\sigma^{2^*-3}(\overline{\Psi})^2+(2^*-1)V_\sigma^{2^*-2}\partial_\sigma\overline{\Psi})\partial_\sigma P\overline{\Psi}+o(1))\\ &= O(\varepsilon^{\frac{2}{N-2}}\sigma^{-4}),\end{aligned}$$

which yields (A.26). □

Lemma A.7. *As* $\mu\to 0^+$,

$$\begin{aligned}&\int_\Omega |\nabla PV_\sigma|^2-\mu\frac{|PV_\sigma|^2}{|x|^2} \qquad\qquad \text{(A.27)}\\ =\ & S_\mu^{\frac{N}{2}}-C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\mu\sigma^{N-2})+O(\sigma^N);\end{aligned}$$

$$\begin{aligned}&\int_\Omega \nabla PV_\sigma\nabla PU_{\delta_i,\xi_i}-\mu\frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2} \qquad\qquad \text{(A.28)}\\ =\ & C_0C_\mu^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\mu\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}});\end{aligned}$$

$$\mu\int_\Omega\frac{|PU_{\delta_i,\xi_i}|^2}{|x|^2}=\mu\frac{C_0^2\delta_i^2}{|\xi_i|^2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{N-2}}+O(\mu\delta_i^4); \qquad \text{(A.29)}$$

$$\mu\int_\Omega\frac{PU_{\delta_i,\xi_i}PU_{\delta_j,\xi_j}}{|x|^2}=O(\mu\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}); \qquad \text{(A.30)}$$

$$\int_\Omega|\nabla PU_{\delta_i,\xi_i}|^2=S_0^{\frac{N}{2}}-C_0^{2^*}H(\xi_i,\xi_i)\delta_i^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\delta_i^{N-2}); \qquad \text{(A.31)}$$

$$\int_\Omega\nabla PU_{\delta_i,\xi_i}\nabla PU_{\delta_j,\xi_j}=C_0^{2^*}G(\xi_i,\xi_j)\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}), \qquad \text{(A.32)}$$

where $i,j=1,2,\dots,k, i\neq j$.

Proof. The proofs of (A.31) and (A.32) are from [2]. We prove the remaining.

(1). Proof of (A.27).

Integration by parts yields

$$\begin{aligned}&\int_\Omega|\nabla PV_\sigma|^2-\mu\frac{|PV_\sigma|^2}{|x|^2}=\int_\Omega(-\Delta V_\sigma)PV_\sigma-\mu\frac{|PV_\sigma|^2}{|x|^2}\\ =\ &\int_\Omega V_\sigma^{2^*-1}PV_\sigma+\mu\frac{V_\sigma PV_\sigma-|PV_\sigma|^2}{|x|^2}\\ =\ &\int_\Omega V_\sigma^{2^*}-\int_\Omega V_\sigma^{2^*-1}\varphi_\sigma+\mu\int_\Omega\frac{\varphi_\sigma(V_\sigma-\varphi_\sigma)}{|x|^2}.\end{aligned}$$

As (B.19) in [2], to continue, let us first show the following.

$$\begin{aligned}
&\int_\Omega V_\sigma^{2^*-1}H(0,x) &&(A.33)\\
=\ &\int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1}H(0,x)+O(\sigma^{\frac{N+2}{2}})\\
=\ &H(0,0)\int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1}+O(\sigma^{\frac{N+2}{2}})\\
&\text{(expanding } H(0,x) \text{ near } x=0)\\
=\ &H(0,0)C_\mu^{2^*-1}\int_{B(0,\frac{\eta}{2})}\frac{\sigma^{\frac{N+2}{2}}}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}}+O(\sigma^{\frac{N+2}{2}})\\
=\ &H(0,0)C_\mu^{2^*-1}\int_{B(0,\frac{\eta}{2}\cdot\sigma^{-\frac{\sqrt{\mu}}{\sqrt{\overline{\mu}-\mu}}})}\frac{\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\sigma^{\frac{N+2}{2}})\quad (\sigma^{-\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}}x=z)\\
=\ &H(0,0)C_\mu^{2^*-1}\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\sigma^{\frac{N+2}{2}}).
\end{aligned}$$

So, by using (2.6),

$$\begin{aligned}
\int_\Omega V_\sigma^{2^*-1}\varphi_\sigma &= C_0C_\mu^{2^*-1}H(0,0)\sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} &&(A.34)\\
&\quad +O(\mu\sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}})+O(\sigma^N)\\
&= C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}+O(\mu\sigma^{N-2})+O(\sigma^N).
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&|\int_\Omega\frac{d^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}(x)H(0,x)}{|x|^2}(\frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}-\frac{1}{|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}})| &&(A.35)\\
\le\ & C\int_{B(0,\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}})}|\frac{1}{|x|^2}(\frac{|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}-(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}})|\\
&+C\int_{\Omega\setminus B(0,\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}})}|\frac{1}{|x|^2}(\frac{|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}-(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}})|\\
\le\ & C\int_{B(0,\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}})}|\frac{1}{|x|^2}\frac{\sigma^{N-2}|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}}{\sigma^{N-2}|x|^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}}|\\
&+C\int_{\Omega\setminus B(0,\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}})}|\frac{1}{|x|^2}\frac{|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}\sigma^2|x|^{\frac{-2\sqrt{\overline{\mu}-\mu}}{\sqrt{\overline{\mu}}}}}{|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}}|\\
=\ & O(\sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}})+O(\sigma^2),
\end{aligned}$$

then

$$\begin{aligned}
\mu\int_\Omega\frac{\varphi_\sigma V_\sigma}{|x|^2} &= (1+o(1))\mu C_\mu^2\sigma^{N-2}\int_\Omega\frac{d^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}(x)H(0,x)}{|x|^2|x|^{\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu}}} &&(A.36)\\
&= O(\mu\sigma^{N-2}).
\end{aligned}$$

It is also easy to see that

$$\mu\int_\Omega\frac{\varphi_\sigma^2}{|x|^2}=O(\mu\sigma^{N-2}), \qquad (A.37)$$

$$\int_\Omega V_\sigma^{2^*} = S_\mu^{\frac{N}{2}} + O(\sigma^N). \tag{A.38}$$

Hence (A.34), (A.36), (A.37), (A.38) yield (A.27).

(2). Proof of (A.28).

By using (2.7), integration by parts yields

$$\begin{aligned}
&\int_\Omega \nabla PV_\sigma \nabla PU_{\delta_i,\xi_i} - \mu \frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2} \\
= &\int_\Omega V_\sigma^{2^*-1} U_{\delta_i,\xi_i} - \int_\Omega V_\sigma^{2^*-1}\varphi_{\delta_i,\xi_i} + \mu \int_\Omega \frac{\varphi_\sigma(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i})}{|x|^2}.
\end{aligned}$$

$$\begin{aligned}
\int_\Omega V_\sigma^{2^*-1}U_{\delta_i,\xi_i} &= \left(\int_{B(0,\frac{\eta}{4})} + \int_{B(\xi_i,\frac{\eta}{4})} + \int_{\Omega\setminus B(0,\frac{\eta}{4})\cup B(\xi_i,\frac{\eta}{4})}\right) V_\sigma^{2^*-1}U_{\delta_i,\xi_i} \\
&= \left(\int_{B(0,\frac{\eta}{4})} + \int_{B(\xi_i,\frac{\eta}{4})}\right) V_\sigma^{2^*-1}U_{\delta_i,\xi_i} + O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned} \tag{A.39}$$

As $\mu \to 0^+$,

$$\begin{aligned}
&\int_{B(0,\frac{\eta}{4})} V_\sigma^{2^*-1}U_{\delta_i,\xi_i} = C_0 C_\mu^{2^*-1}\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}} \cdot \frac{1}{(\delta_i^2 + |x-\xi_i|^2)^{\frac{N-2}{2}}} \\
= \ & C_0 C_\mu^{2^*-1}\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}} \left(\frac{1}{(\delta_i^2+|\xi_i|^2)^{\frac{N-2}{2}}} + O(|x|^2)\right) \\
= \ & C_0 C_\mu^{2^*-1}\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{|\xi_i|^{N-2}} + o(\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}}),
\end{aligned}$$

$$\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{4})} V_\sigma^{2^*-1}U_{\delta_i,\xi_i} = C_0 C_\mu^{2^*-1}\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{4})} \frac{1}{(\sigma^2|x+\xi_i|^{\beta_1}+|x+\xi_i|^{\beta_2})^{\frac{N+2}{2}}} \cdot \frac{1}{(\delta_i^2+|x|^2)^{\frac{N-2}{2}}} \\
\leq \ & O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}),
\end{aligned}$$

then

$$(A.39) \quad = \quad C_0 C_\mu^{2^*-1}\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{|\xi_i|^{N-2}} + o(\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}}).$$

On the other hand,

$$\begin{aligned}
&\int_\Omega V_\sigma^{2^*-1}\varphi_{\delta_i,\xi_i} \\
= \ & \int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1}\varphi_{\delta_i,\xi_i} + O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}) \\
= \ & C_0 C_\mu^{2^*-1}\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{2})} \frac{H(\xi_i,x)}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}) \\
= \ & C_0 C_\mu^{2^*-1}\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}} \int_{B(0,\frac{\eta}{2})} \frac{H(\xi_i,0)}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}} + O(\sigma^{\frac{N+2}{2}}\delta_i^{\frac{N-2}{2}}) \\
= \ & C_0 C_\mu^{2^*-1}\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{H(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} + o(\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}}\delta_i^{\frac{N-2}{2}}).
\end{aligned} \tag{A.40}$$

Similarly to (A.35), we have

$$\left|\int_\Omega \frac{d^{\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-\mu}}(x)H(0,x)}{|x|^2}\left(\frac{1}{(\delta_i^2+|x-\xi_i|^2)^{\frac{N-2}{2}}} - \frac{1}{(|x-\xi_i|^2)^{\frac{N-2}{2}}}\right)\right| \leq O(\delta_i^2), \tag{A.41}$$

which yields

$$\begin{aligned}
&\mu\int_\Omega \frac{\varphi_\sigma U_{\delta_i,\xi_i}}{|x|^2} &(A.42)\\
=\ &\mu C_\mu C_0 \sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_\Omega \frac{\overline{d}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}(x)H(0,x)}{|x|^2|x-\xi_i|^{N-2}}+o(\mu\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})\\
=\ &O(\mu\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned}$$

It is also easy to see that

$$\mu\int_\Omega \frac{\varphi_\sigma\varphi_{\delta_i,\xi_i}}{|x|^2}=O(\mu\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}). \tag{A.43}$$

By (A.39), (A.40), (A.42), (A.43), we conclude (A.28).

(3). Proof of (A.29).

$$\begin{aligned}
&\int_\Omega \frac{U^2_{\delta_i,\xi_i}}{|x|^2} &(A.44)\\
=\ &C_0^2\delta_i^{N-2}\int_\Omega \frac{1}{|x|^2(\delta_i^2+|x-\xi_i|^2)^{N-2}}=C_0^2\delta_i^{N-2}(C+\int_{B(\xi_i,\frac{\eta}{4})}\frac{1}{|x|^2(\delta_i^2+|x-\xi_i|^2)^{N-2}})\\
=\ &C_0^2\delta_i^{N-2}(C+\int_{B(0,\frac{\eta}{4})}\frac{1}{|x+\xi_i|^2(\delta_i^2+|x|^2)^{N-2}})=C_0^2\delta_i^{N-2}(C+\int_{B(0,\frac{\eta}{4})}\frac{1+O(|x|^2)}{|\xi_i|^2(\delta_i^2+|x|^2)^{N-2}})\\
=\ &C_0^2\delta_i^{N-2}(C+\frac{1}{|\xi_i|^2}(-\int_{\mathbb{R}^N\setminus B(0,\frac{\eta}{4\delta_i})}+\int_{\mathbb{R}^N})\frac{\delta_i^{4-N}}{(1+|z|^2)^{N-2}}+\frac{1}{|\xi_i|^2}\int_{B(0,\frac{\eta}{4})}\frac{O(|x|^2)}{(\delta_i^2+|x|^2)^{N-2}})\\
=\ &\frac{C_0^2}{|\xi_i|^2}\delta_i^2\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{N-2}}+O(\delta_i^4).
\end{aligned}$$

On the other hand,

$$\int_\Omega \frac{\varphi^2_{\delta_i,\xi_i}}{|x|^2} = O(\delta_i^{N-2}); \tag{A.45}$$

$$\int_\Omega \frac{\varphi_{\delta_i,\xi_i}U_{\delta_i,\xi_i}}{|x|^2} = O(\delta_i^{N-2}). \tag{A.46}$$

Then (A.44), (A.45), (A.46) yield (A.29).

(4). Proof of (A.30).

We omit it here since it is similarly to (A.29). □

Lemma A.8. *As* $\mu \to 0^+$,

$$\int_\Omega |PV_\sigma|^{2^*} = S_\mu^{\frac{N}{2}} - 2^* C_0 C_\mu^{2^*-1} H(0,0)\sigma^{N-2} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} + O(\mu\sigma^{N-2}) + O(\sigma^N); \tag{A.47}$$

$$\int_\Omega |PU_{\delta_i,\xi_i}|^{2^*} = S_0^{\frac{N}{2}} - 2^* C_0^{2^*} H(\xi_i,\xi_i)\delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} + O(\delta_i^N); \tag{A.48}$$

$$\int_\Omega |-\sum_{i=1}^k PU_{\delta_i,\xi_i} + PV_\sigma|^{2^*} \tag{A.49}$$

$$\begin{aligned}
&= S_\mu^{\frac{N}{2}} - 2^* C_0 C_\mu^{2^*-1} H(0,0)\sigma^{N-2} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\quad -2^* \sum_{i=1}^k C_0 C_\mu^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\quad +\sum_{i=1}^k [S_0^{\frac{N}{2}} - 2^* C_0^{2^*} H(\xi_i,\xi_i)\delta_i^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \\
&\quad +2^* \sum_{j=1, j\neq i}^k C_0^{2^*} \delta_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i,\xi_j)}{(1+|z|^2)^{\frac{N+2}{2}}} \\
&\quad -2^* C_\mu C_0^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i,0)}{(1+|z|^2)^{\frac{N+2}{2}}}] \\
&\quad +\sum_{i=1}^k (o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}) + \sum_{j=1, j\neq i}^k o(\delta_j^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}) + O(\delta_i^N)) + O(\mu\sigma^{N-2}) + O(\sigma^N).
\end{aligned}$$

Proof. (A.48) is from [2].

By (A.34),

$$\begin{aligned}
\int_\Omega |PV_\sigma|^{2^*} &= \int_\Omega V_\sigma^{2^*} - 2^* \int_\Omega V_\sigma^{2^*-1}\varphi_\sigma + O(\sigma^N) \\
&= S_\mu^{\frac{N}{2}} - 2^* C_0 C_\mu^{2^*-1} H(0,0)\sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}} \int_{\mathbb{R}^N} \frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} \\
&\quad +O(\mu\sigma^{\frac{\sqrt{\overline{\mu}}(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}}) + O(\sigma^N).
\end{aligned}$$

Now we turn to (A.49). By (2.6), (2.7), (A.39), (A.40),

$$\int_{B(0,\frac{\eta}{2})} (PV_\sigma)^{2^*-1} PU_{\delta_i,\xi_i} = \int_{B(0,\frac{\eta}{2})} (V_\sigma^{2^*-1} + O(V_\sigma^{2^*-2}\varphi_\sigma))(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i}) \tag{A.50}$$

$$\begin{aligned}
&= \int_{B(0,\frac{\eta}{2})} V_\sigma^{2^*-1}(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i}) + O(\sigma^{\frac{\sqrt{\overline{\mu}}(N-2+\sqrt{\overline{\mu}-\mu})}{\sqrt{\overline{\mu}-\mu}}} \delta_i^{\frac{N-2}{2}}) \\
&= C_0 C_\mu^{2^*-1} \sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{G(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}} + o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned}$$

Then by (A.47),

$$
\begin{aligned}
&\int_{B(0,\frac{\eta}{2})}|-\sum_{i=1}^{k}PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*} \qquad\qquad (A.51)\\
=\ &\int_{B(0,\frac{\eta}{2})}(PV_\sigma)^{2^*}-2^*\sum_{i=1}^{k}\int_{B(0,\frac{\eta}{2})}(PV_\sigma)^{2^*-1}PU_{\delta_i,\xi_i}+\sum_{i=1}^{k}O(\int_{B(0,\frac{\eta}{2})}(PV_\sigma)^{2^*-2}(PU_{\delta_i,\xi_i})^2)\\
=\ &S_\mu^{\frac{N}{2}}-2^*C_0C_\mu^{2^*-1}H(0,0)\sigma^{N-2}\int_{\mathbb{R}^N}\frac{1}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}\\
&-2^*\sum_{i=1}^{k}C_0C_\mu^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,0)}{(|z|^{\beta_1}+|z|^{\beta_2})^{\frac{N+2}{2}}}\\
&+\sum_{i=1}^{k}o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})+O(\mu\sigma^{N-2})+O(\sigma^N).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{2})}|-\sum_{i=1}^{k}PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*} \qquad\qquad (A.52)\\
=\ &\int_{B(\xi_i,\frac{\eta}{2})}|PU_{\delta_i,\xi_i}+\sum_{j=1,j\neq i}^{k}PU_{\delta_j,\xi_j}-PV_\sigma|^{2^*}\\
=\ &\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*}+2^*\sum_{j=1,j\neq i}^{k}(PU_{\delta_i,\xi_i})^{2^*-1}PU_{\delta_j,\xi_j}-2^*(PU_{\delta_i,\xi_i})^{2^*-1}PV_\sigma\\
&+O(\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*-2}(-\sum_{j=1,j\neq i}^{k}PU_{\delta_j,\xi_j}+PV_\sigma)^2)\\
=\ &S_0^{\frac{N}{2}}-2^*C_0^{2^*}H(\xi_i,\xi_i)\delta_i^{N-2}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}+O(\delta_i^N)\\
&+2^*\sum_{j=1,j\neq i}^{k}C_0^{2^*}\delta_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,\xi_j)}{(1+|z|^2)^{\frac{N+2}{2}}}+\sum_{j=1,j\neq i}^{k}o(\delta_j^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})\\
&-2^*C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,0)}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}),
\end{aligned}
$$

where the last equality was obtained by the results in [2] and

$$
\begin{aligned}
&\int_{B(\xi_i,\frac{\eta}{2})}(PU_{\delta_i,\xi_i})^{2^*-1}PV_\sigma \qquad\qquad (A.53)\\
=\ &\int_{B(\xi_i,\frac{\eta}{2})}U_{\delta_i,\xi_i}^{2^*-1}(V_\sigma-\varphi_\sigma)+o(\delta_i^{\frac{N-2}{2}}\sigma^{\frac{N-2}{2}})\\
=\ &C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}(\frac{1}{|\xi_i|^{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}}-\overline{d}^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}(x)H(0,\xi_i))+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}})\\
=\ &C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{G(\xi_i,0)}{(1+|z|^2)^{\frac{N+2}{2}}}+o(\sigma^{\frac{N-2}{2}}\delta_i^{\frac{N-2}{2}}).
\end{aligned}
$$

At last,

$$
\int_{\Omega\setminus B(0,\frac{\eta}{2})\bigcup\bigcup\limits_{i=1}^{k}B(\xi_i,\frac{\eta}{2})}|-\sum_{i=1}^{k}PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*}\le\sum_{i=1}^{k}O(\delta_i^N)+O(\sigma^N). \qquad (A.54)
$$

Then (A.51),(A.52),(A.54) yield (A.49). □

Lemma A.9.

$$\begin{aligned}&\int_\Omega |-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*}\ln|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma| \\ = \;& -\frac{N-2}{2}\ln\sigma\cdot\int_{\mathbb{R}^N}V_1^{2^*}-\frac{N-2}{2}\ln(\delta_1\delta_2\ldots\delta_k)\cdot\int_{\mathbb{R}^N}U_{1,0}^{2^*} \\ & +\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1+k\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}+o(1).\end{aligned} \tag{A.55}$$

Proof. Similarly to [13],

$$\begin{aligned}&\int_{B(0,\frac{\eta}{2})}|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*}\ln|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma| \\ = \;& -\frac{\overline{\mu}}{\sqrt{\overline{\mu}-\mu}}\ln\sigma\cdot\int_{\mathbb{R}^N}V_1^{2^*}+\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1+o(1) \\ = \;& -\frac{N-2}{2}\ln\sigma\cdot\int_{\mathbb{R}^N}V_1^{2^*}+\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1+o(1),\end{aligned} \tag{A.56}$$

$$\begin{aligned}&\int_{B(\xi_i,\frac{\eta}{2})}|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*}\ln|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma| \\ = \;& -\frac{N-2}{2}\ln\delta_i\cdot\int_{\mathbb{R}^N}U_{1,0}^{2^*}+\int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}+o(1),\end{aligned} \tag{A.57}$$

$$\begin{aligned}&\int_{\Omega\setminus B(0,\frac{\eta}{2})\bigcup\limits_{i=1}^k B(\xi_i,\frac{\eta}{2})}|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma|^{2^*}\ln|-\sum_{i=1}^k PU_{\delta_i,\xi_i}+PV_\sigma| \\ = \;& o(1),\end{aligned} \tag{A.58}$$

then we conclude (A.55).

□

Lemma A.10. *As* $\mu\to 0^+$,

$$\int_{\mathbb{R}^N}V_1^{2^*-1} = \int_{\mathbb{R}^N}U_{1,0}^{2^*-1}+o(1); \tag{A.59}$$

$$\int_{\mathbb{R}^N}V_1^{2^*} = \int_{\mathbb{R}^N}U_{1,0}^{2^*}+o(1); \tag{A.60}$$

$$\int_{\mathbb{R}^N}V_1^{2^*}\ln V_1 = \int_{\mathbb{R}^N}U_{1,0}^{2^*}\ln U_{1,0}+o(1); \tag{A.61}$$

$$C_\mu = C_0-\frac{C_0}{N-2}\mu+O(\mu^2); \tag{A.62}$$

$$S_\mu = S_0-\overline{S}\mu+O(\mu^2), \tag{A.63}$$

for some positive constant $\overline{S}$ *independent of* μ.

Proof. The equalities can be obtained by direct computations. □

B Appendix

The lemmas used in Section 4 are listed below.

As Lemma A.1, we have the following.

Lemma B.1. *For $i, l = 1, 2, \ldots, k$, and $j, h = 0, 1, \ldots, N$, it holds*

$$\begin{aligned}
(P\overline{\Psi}, P\overline{\Psi}) &= \widetilde{c}_0 \frac{1}{\sigma^2} + o(\frac{1}{\sigma^2}); && \text{(B.1)}\\
(P\overline{\Psi}, P\Psi_i^j) &= o(\frac{1}{\sigma^2})(\textit{and } o(\frac{1}{\delta_i^2})); && \text{(B.2)}\\
(P\Psi_i^j, P\Psi_i^j) &= \widetilde{c}_{i,j} \frac{1}{\delta_i^2} + o(\frac{1}{\delta_i^2}); && \text{(B.3)}\\
(P\Psi_i^j, P\Psi_l^h) &= o(\frac{1}{\delta_i^2})(\textit{and } o(\frac{1}{\delta_l^2})) \quad \textit{if } i \neq l \textit{ or } j \neq h, && \text{(B.4)}
\end{aligned}$$

where $\widetilde{c}_0 > 0, \widetilde{c}_{i,j} > 0$ are constants.

Lemma B.2.

$$\begin{aligned}
\|(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h\|_{2N/(N+2)} &= o(\frac{1}{\delta_l^{\frac{2N}{N+2}}}), && \text{(B.5)}\\
\|(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(V_\sigma))\overline{\Psi}\|_{2N/(N+2)} &= o(\frac{1}{\sigma^{\frac{2N}{N+2}}}). && \text{(B.6)}
\end{aligned}$$

Proof. We only prove (B.5) for $h \neq 0$.

$$\begin{aligned}
&\int_\Omega |(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} && \text{(B.7)}\\
= &\bigcup_{i=1}^{k+1}\int_{A_i} |(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)}\\
&+ \int_{\Omega\setminus B(0,\rho)} |(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)}.
\end{aligned}$$

As Lemma A.3 in [25], by (2.6), (2.7),

$$\begin{aligned}
&\int_{A_l} |(f_0'(\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} && \text{(B.8)}\\
\le\ & C\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3}\varphi_{\delta_l,\xi_l}\Psi_l^h|^{2N/(N+2)} + C\sum_{i\neq l}\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3}U_{\delta_i,\xi_i}\Psi_l^h|^{2N/(N+2)}\\
&+C\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3}V_\sigma\Psi_l^h|^{2N/(N+2)}\\
\le\ & o(\frac{1}{\delta_l^{\frac{2N}{N+2}}}),
\end{aligned}$$

since

$$\begin{aligned}
&\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3}\varphi_{\delta_l,\xi_l}\Psi_l^h|^{2N/(N+2)}\\
\le\ & C\int_{A_l} |\frac{\delta_l^{\frac{N+2}{2}}(x^h - \xi_l^h)}{(\delta_l^2 + |x-\xi_l|^2)^3}|^{2N/(N+2)} = O(\delta_l^{\frac{2N(N-3)}{N+2}}),
\end{aligned}$$

for $i \neq l$,

$$\begin{aligned}
&\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3} U_{\delta_i,\xi_i} \Psi_l^h|^{2N/(N+2)} \\
= \;& C\int_{A_l} |\frac{\delta_l^2(x^h-\xi_l^h)}{(\delta_l^2+|x-\xi_l|^2)^3} \frac{\delta_i^{\frac{N-2}{2}}}{(\delta_i^2+|x-\xi_i|^2)^{\frac{N-2}{2}}}|^{2N/(N+2)} \\
\leq \;& C(\int_{A_l} |\frac{\delta_l^2(x^h-\xi_l^h)}{(\delta_l^2+|x-\xi_l|^2)^3}|^{\frac{N}{2}})^{\frac{4}{N+2}} (\int_{A_l} |\frac{\delta_i^{\frac{N-2}{2}}}{(\delta_i^2+|x-\xi_i|^2)^{\frac{N-2}{2}}}|^{2N/(N-2)})^{\frac{N-2}{N+2}} \\
= \;& o(\frac{1}{\delta_l^{\frac{2N}{N+2}}}),
\end{aligned}$$

and similarly,

$$\int_{A_l} |U_{\delta_l,\xi_l}^{2^*-3} V_\sigma \Psi_l^h|^{2N/(N+2)} = o(\frac{1}{\delta_l^{\frac{2N}{N+2}}}).$$

The same arguments as (B.8) give that, for $i \neq l$,

$$\int_{A_i} |(f_0'(\sum_{i=1}^k (-1)^{i-1} PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} = o(\frac{1}{\delta_l^{\frac{2N}{N+2}}}). \tag{B.9}$$

At last,

$$\begin{aligned}
&\int_{\Omega\setminus B(0,\rho)} |(f_0'(\sum_{i=1}^k (-1)^{i-1} PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma) - f_0'(U_{\delta_l,\xi_l}))\Psi_l^h|^{2N/(N+2)} \\
= \;& \begin{cases} O(\delta_l^{\frac{N(N-2)}{N+2}})(O(\sigma^{\frac{4N}{N+2}}) + \sum_{i=1}^k O(\delta_i^{\frac{4N}{N+2}})) & \text{if } \; h = 1,2,\ldots,N, \\ O(\delta_l^{\frac{N(N-4)}{N+2}})(O(\sigma^{\frac{4N}{N+2}}) + \sum_{i=1}^k O(\delta_i^{\frac{4N}{N+2}})) & \text{if } \; h = 0. \end{cases}
\end{aligned} \tag{B.10}$$

Then (B.5) follows. □

Lemma B.3.

$$\|\iota^*(\sum_{i=1}^k (-1)^{i-1} f_0(U_{\delta_i,\xi_i}) + (-1)^k f_0(V_\sigma)) - V_{\varepsilon,\lambda,\xi}\|_\mu \leq \sum_{i=1}^k O(\mu\delta_i) + O((\mu\sigma^{\frac{N-2}{2}})^{\frac{1}{2}}). \tag{B.11}$$

Proof. It is similarly to Lemma A.4. □

Lemma B.4.

$$\|(f_\varepsilon'(V_{\varepsilon,\lambda,\xi}) - f_0'(V_{\varepsilon,\lambda,\xi}))\phi\|_{2N/(N+2)} = C\varepsilon\|\phi\|_\mu; \tag{B.12}$$

$$\|f_\varepsilon(V_{\varepsilon,\lambda,\xi}) - f_0(V_{\varepsilon,\lambda,\xi})\|_{2N/(N+2)} = C\varepsilon; \tag{B.13}$$

$$\|f_0(V_{\varepsilon,\lambda,\xi}) - (\sum_{i=1}^k (-1)^{i-1} f_0(U_{\delta_i,\xi_i}) + (-1)^k f_0(V_\sigma))\|_{2N/(N+2)} = O(\varepsilon^{\frac{N+2}{2(N-2)}}). \tag{B.14}$$

Proof. The first two, as Lemma A.5, are from [4]. The last one can be proved as (4.5) in [25]. □

Lemma B.5. *Let $k \geq 1$. Then*

$$\int_\Omega |\nabla PV_\sigma|^2 - \mu \frac{|PV_\sigma|^2}{|x|^2} = S_\mu^{\frac{N}{2}} + o(\varepsilon); \tag{B.15}$$

$$\int_\Omega \nabla PV_\sigma \nabla PU_{\delta_i,\xi_i} - \mu \frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2} \tag{B.16}$$

$$= \begin{cases} C_0^{2^*}(\frac{\lambda}{\lambda_k})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{(1+|\zeta_k|^2)^{\frac{N-2}{2}}} \cdot \varepsilon + o(\varepsilon) & \textit{if } \ i = k, \\ o(\varepsilon) & \textit{if } \ i \neq k; \end{cases}$$

$$\mu \int_\Omega \frac{|PU_{\delta_i,\xi_i}|^2}{|x|^2} = \mu C_0^2 \int_{\mathbb{R}^N} \frac{1}{|y|^2(1+|y-\zeta_i|^2)^{N-2}} + o(\varepsilon); \tag{B.17}$$

$$\mu \int_\Omega \frac{PU_{\delta_i,\xi_i} PU_{\delta_j,\xi_j}}{|x|^2} = o(\varepsilon), \quad i \neq j; \tag{B.18}$$

$$\int_\Omega |\nabla PU_{\delta_i,\xi_i}|^2 \tag{B.19}$$

$$= \begin{cases} S_0^{\frac{N}{2}} - C_0^{2^*} H(0,0) \lambda_1^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \cdot \varepsilon + o(\varepsilon) & \textit{if } \ i = 1, \\ S_0^{\frac{N}{2}} + o(\varepsilon) & \textit{if } \ i \neq 1; \end{cases}$$

$$\int_\Omega \nabla PU_{\delta_i,\xi_i} \nabla PU_{\delta_j,\xi_j} \tag{B.20}$$

$$= \begin{cases} C_0^{2^*}(\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{(1+|\zeta_i|^2)^{\frac{N-2}{2}}} \cdot \varepsilon + o(\varepsilon) & \textit{if } \ j = i+1, \\ o(\varepsilon) & \textit{otherwise}, \end{cases}$$

where we assume, without loss of generality, $1 \leq i < j \leq k$.

Proof. (B.15) and (B.19) can be obtained by (A.27) and (A.31), respectively.

(1). Proof of (B.16).

By using (2.7), integration by parts yields

$$\int_\Omega \nabla PV_\sigma \nabla PU_{\delta_i,\xi_i} - \mu \frac{PV_\sigma PU_{\delta_i,\xi_i}}{|x|^2} = \int_\Omega V_\sigma^{2^*-1}(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i}) + \mu \int_\Omega \frac{\varphi_\sigma(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i})}{|x|^2}. \tag{B.21}$$

It is easy to show, by using (2.6) and (2.7), that

$$\begin{aligned} \int_\Omega V_\sigma^{2^*-1} \varphi_{\delta_i,\xi_i} \quad &\leq \quad C\delta_i^{\frac{N-2}{2}} \int_\Omega \frac{\sigma^{\frac{N+2}{2}}}{(\sigma^2|x|^{\beta_1} + |x|^{\beta_2})^{\frac{N+2}{2}}} \\ &\leq \quad C\sigma^{\frac{\mu}{\sqrt{\overline{\mu}-\mu}}} \delta_i^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(|y|^{\beta_1} + |y|^{\beta_2})^{\frac{N+2}{2}}} = O(\sigma^{\frac{N-2}{2}} \delta_i^{\frac{N-2}{2}}), \end{aligned} \tag{B.22}$$

and

$$\mu \int_\Omega \frac{\varphi_\sigma(U_{\delta_i,\xi_i} - \varphi_{\delta_i,\xi_i})}{|x|^2} \quad \leq \quad \mu \int_\Omega \frac{\varphi_\sigma U_{\delta_i,\xi_i}}{|x|^2} \leq O(\mu \sigma^{\frac{N-2}{2}}). \tag{B.23}$$

On the other hand,

$$\int_\Omega V_\sigma^{2^*-1} U_{\delta_i,\xi_i} = \bigcup_{j=1}^{k+1} \int_{A_j} V_\sigma^{2^*-1} U_{\delta_i,\xi_i} + O(\sigma^{\frac{N+2}{2}} \delta_i^{\frac{N-2}{2}}). \tag{B.24}$$

If $i=k, j=k+1$,

$$\begin{aligned}
&\int_{A_{k+1}} V_\sigma^{2^*-1} U_{\delta_k,\xi_k} \qquad \text{(B.25)}\\
=\ & C_\mu^{2^*-1} C_0 \sigma^{\frac{N+2}{2}} \delta_k^{\frac{N-2}{2}} \int_{A_{k+1}} \frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{(\delta_k^2+|x-\xi_k|^2)^{\frac{N-2}{2}}}\\
=\ & C_\mu^{2^*-1} C_0 \frac{\sigma^{\frac{\overline{\mu}}{\sqrt{\overline{\mu}}-\mu}}}{\delta_k^{\frac{N-2}{2}}} \int_{\frac{A_{k+1}}{\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}}}} \frac{1}{(|y|^{\beta_1}+|y|^{\beta_2})^{\frac{N+2}{2}}} \frac{1}{(1+|\frac{\sigma^{\frac{\sqrt{\overline{\mu}}}{\sqrt{\overline{\mu}-\mu}}}}{\delta_k} y-\zeta_k|^2)^{\frac{N-2}{2}}}\\
=\ & C_0^{2^*} (\frac{\overline{\lambda}}{\lambda_k})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{(1+|\zeta_k|^2)^{\frac{N-2}{2}}} \cdot \varepsilon + o(\varepsilon)
\end{aligned}$$

If $i \neq k$ or $j \neq k+1$, similar arguments as (B.25) give

$$\int_{A_j} V_\sigma^{2^*-1} U_{\delta_i,\xi_i} = o(\varepsilon). \qquad \text{(B.26)}$$

Then we conclude by (B.21)-(B.26).

(2). Proof of (B.17).

$$\begin{aligned}
(B.17) &= \mu \int_\Omega \frac{|U_{\delta_i,\xi_i}|^2}{|x|^2} + O(\mu\delta_i^{N-2})\\
&= \mu C_0^2 \int_{\frac{\Omega}{\delta_i}} \frac{1}{|y|^2(1+|y-\zeta_i|^2)^{N-2}} + O(\mu\delta_i^{N-2})\\
&= \mu C_0^2 \int_{\mathbb{R}^N} \frac{1}{|y|^2(1+|y-\zeta_i|^2)^{N-2}} + o(\varepsilon).
\end{aligned}$$

Similar arguments give that $(B.18) = o(\varepsilon)$

(3). Proof of (B.20).

Without loss of generality, let $1 \leq i < j \leq k$. Then as (B.24),

$$\begin{aligned}
(B.20) &= \int_\Omega U_{\delta_j,\xi_j}^{2^*-1} U_{\delta_i,\xi_i} + o(\varepsilon) \qquad \text{(B.27)}\\
&= \begin{cases} C_0^{2^*} (\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{(1+|\zeta_i|^2)^{\frac{N-2}{2}}} \cdot \varepsilon + o(\varepsilon) & \text{if } \ j = i+1,\\ o(\varepsilon) & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Lemma B.6. *Let $k \geq 1$. Then*

$$\begin{aligned}
&\int_\Omega |\sum_{i=1}^k (-1)^{i-1} PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma|^{2^*} \qquad \text{(B.28)}\\
=\ & kS_0^{\frac{N}{2}} + S_\mu^{\frac{N}{2}} - 2^* C_0^{2^*} H(0,0) \lambda_1^{N-2} \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} \cdot \varepsilon\\
& -2^* C_0^{2^*} \sum_{i=1}^k (\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{|y|^{N-2}(1+|y-\zeta_i|^2)^{\frac{N+2}{2}}} \cdot \varepsilon\\
& -2^* C_0^{2^*} \sum_{i=1}^k (\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2}{2}}} \frac{1}{(1+|\zeta_i|^2)^{\frac{N-2}{2}}} \cdot \varepsilon + o(\varepsilon),
\end{aligned}$$

where $\lambda_{k+1} = \overline{\lambda}$.

Proof.

$$(B.28) = \bigcup_{j=1}^{k+1} \int_{A_j} |\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^k PV_\sigma|^{2^*} + O(\delta_1^N). \tag{B.29}$$

First of all,

$$\begin{aligned}
&\int_{A_k} V_\sigma U_{\delta_k,\xi_k}^{2^*-1} \qquad\qquad (B.30)\\
= \ & C_\mu C_0^{2^*-1}\sigma^{\frac{N-2}{2}}\delta_k^{\frac{N+2}{2}} \int_{A_k} \frac{1}{(\sigma^2|x|^{\beta_1}+|x|^{\beta_2})^{\frac{N-2}{2}}} \frac{1}{(\delta_k^2+|x-\xi_k|^2)^{\frac{N+2}{2}}}\\
= \ & C_\mu C_0^{2^*-1} \frac{\sigma^{\frac{N-2}{2}}}{\delta_k^{\sqrt{\bar\mu-\mu}}} \int_{\frac{A_k}{\delta_k}} \frac{1}{((\frac{\sigma}{\delta_k})^2|y|^{\beta_1}+|y|^{\beta_2})^{\frac{N-2}{2}}} \frac{1}{(1+|y-\zeta_k|^2)^{\frac{N+2}{2}}}\\
= \ & C_0^{2^*}(\frac{\overline{\lambda}}{\lambda_k})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{|y|^{N-2}(1+|y-\zeta_k|^2)^{\frac{N+2}{2}}}\cdot\varepsilon + o(\varepsilon),
\end{aligned}$$

and

$$\int_{A_j} V_\sigma U_{\delta_i,\xi_i}^{2^*-1} = o(\varepsilon), \text{ if } i\neq k, \text{ or } j\neq k. \tag{B.31}$$

From [25], we also have, for $1\leq i<j\leq k$,

$$\begin{aligned}
&\int_{A_l} U_{\delta_j,\xi_j}U_{\delta_i,\xi_i}^{2^*-1} + o(\varepsilon) \qquad\qquad (B.32)\\
= &\begin{cases} C_0^{2^*}(\frac{\lambda_{i+1}}{\lambda_i})^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \frac{1}{|y|^{N-2}(1+|y-\zeta_i|^2)^{\frac{N+2}{2}}}\cdot\varepsilon + o(\varepsilon) & \text{if } \ j=i+1, i=l,\\ o(\varepsilon) & \text{otherwise.}\end{cases}
\end{aligned}$$

Noticing (B.25), (B.27) and the above three equalities, then the proof of (B.28) is actually involved by Lemma 6.2 in [25]. □

Lemma B.7.

$$\begin{aligned}
&\int_\Omega |\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^kPV_\sigma|^{2^*} \ln|\sum_{i=1}^{k}(-1)^{i-1}PU_{\delta_i,\xi_i} + (-1)^kPV_\sigma| \qquad (B.33)\\
= \ & -\frac{N-2}{2}\ln\sigma\cdot\int_{\mathbb{R}^N} V_1^{2^*} - \frac{N-2}{2}\ln(\delta_1\delta_2\ldots\delta_k)\cdot\int_{\mathbb{R}^N} U_{1,0}^{2^*}\\
&+\int_{\mathbb{R}^N} V_1^{2^*}\ln V_1 + k\int_{\mathbb{R}^N} U_{1,0}^{2^*}\ln U_{1,0} + o(1).
\end{aligned}$$

Proof. The proof is similarly to Lemma A.9. □

References

[1] Th. Aubin, Problèmes isopérimétriques et espaces de Sobolev, J. Differential Geometry 11 (1976), 573-598.

[2] A. Bahri, J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988), 253-290.

[3] A. Bahri, Y. Li, O. Rey, On a variational problem with lack of compactness: the topological effect of the critical points at infinity, Calc. Var. Part. Diff. Equ. 3 (1995), 67-93.

[4] T. Bartsch, A. Micheletti, A. Pistoia, On the existence and the profile of nodal solutions of elliptic equations involving critical growth, Calc. Var. Part. Diff. Equ. 26 (2006), 265-282.

[5] T. Bartsch, T. D'Aprile, A. Pistoia, Multi-bubble nodal solutions for slightly subcritical elliptic problems in domains with symmetries, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013), 1027-1047.

[6] T. Bartsch, T. D'Aprile, A. Pistoia, On the profile of sign-changing solutions of an almost critical problem in the ball, Bull. London Math. Soc. 45 (2013), 1246-1258.

[7] G. Bianchi, H. Egnell, A note on the Sobolev inequality, J. Functional Analysis 100 (1991), 18-24.

[8] H. Brézis, L.A. Peletier, Asymptotics for elliptic equations involving critical growth, Partial differential equations and the calculus of variations, vol. I, Progr. Nonlinear Diff. Equ. Appl. Birkhäuser, Boston, MA 1 (1989), 149-192.

[9] D.M. Cao, P.G. Han, Solutions for semilinear elliptic equations with critical exponents and Hardy potential, J. Diff. Equ. 205 (2004), 521-537.

[10] D.M. Cao, S.J. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, J. Diff. Equ. 193 (2003), 424-434.

[11] D.M. Cao, S.J. Peng, Asymptotic behavior for elliptic problems with singular coefficient and nearly critical Sobolev growth, Annali di Matematica 185 (2006), 189-205.

[12] F. Catrina, Z.Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001), 229-258.

[13] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem, Calc. Var. Part. Diff. Equ. 16(2) (2003), 113-145.

[14] I. Ekeland, N. Ghoussoub, Selected new aspects of the calculus of variations in the large, Bull. Amer. Math. Soc. (N.S.) 39(2) (2002), 207-265.

[15] V. Felli, A. Pistoia, Existence of blowing-up solutions for a nonlinear elliptic equation with Hardy potential and critical growth, Comm. Part. Diff. Equ. 31 (2006), 21-56.

[16] V. Felli, S. Terracini, Fountain-like solutions for nonlinear elliptic equations with critical growth and Hardy potential, Comm. Contemp. Math. 7 (2005), 867-904.

[17] A. Ferrero, F. Gazzola, Existence of solutions for singular critical growth semilinear elliptic equations, J. Diff. Equ. 177 (2001), 494-522.

[18] M. Flucher, J. Wei, Semilinear Dirichlet problem with nearly critical exponent, asymptotic location of hot spots, Manuscripta Math. 94 (1997), 337-346.

[19] N. Ghoussoub, C. Yuan, Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents, Trans. Amer. Math. Soc. 352 (2000), 5703-5743.

[20] M. Grossi, F. Takahashi, Nonexistence of multi-bubble solutions to some elliptic equations on convex domains, J. Functional Analysis 259 (2010), 904-917.

[21] Q.Q. Guo, P.C. Niu, Nodal and positive solutions for singular semilinear elliptic equations with critical exponents in symmetric domains, J. Diff. Equ. 245 (2008), 3974-3985.

[22] Z.C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 8 (1991), 159-174.

[23] E. Jannelli, The role played by space dimension in elliptic critical problems, J. Diff. Equ. 156 (1999), 407-426.

[24] M. Musso, A. Pistoia, Multispike solutions for a nonlinear elliptic problem involving critical Sobolev exponent, Indiana Univ. Math. J. (5) (2002), 541-579.

[25] M. Musso, A. Pistoia, Tower of bubbles for almost critical problems in general domains, J. Math. Pures Appl. 93 (2010), 1-40.

[26] A. Pistoia, T. Weth, Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 325-340.

[27] O. Rey, Proof of two conjectures of H. Brezis and L.A. Peletier, Manuscripta Math. 65 (1989), 19-37.

[28] O. Rey, The role of the Greens function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Functional Analysis 89 (1990), 1-52.

[29] O. Rey, Blow-up points of solutions to elliptic equations with limiting nonlinearity, Differential Integral Equations 4 (1991), 1155-1167.

[30] D. Ruiz, M. Willem, Elliptic problems with critical exponents and Hardy potentials, J. Diff. Equ. 190 (2003), 524-538.

[31] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, Trans. Amer. Math. Soc. 357 (2005), 2909-2938.

[32] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976), 353-372.

[33] S.Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Diff. Equations 2 (1996), 241-264.

[34] Z.Q. Wang, M. Willem, Caffarelli-Kohn-Nirenberg inequalities with remainder terms, J. Functional Analysis 203 (2003), 550-568.